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Nonlinear Wave Equations

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1 Overview

The following notes are a (brief) introduction into the theory of nonlinear wave equations, and mathematical fluids that I've written in preparation for the mathematics PhD oral examination at Stony Brook. These notes cover a few fundamental results on the (local) well-posedness of (non)linear wave equations, energy estimates, Strichartz estimates, and related results in harmonic analysis and fluids. The content reflects my most current understanding of the key ideas, and were written alongside discussions with my advisor, Professor John Anderson. Moreover, these notes are strongly based on the corresponding notes of Professors Holzegel, Lawrie, and Luk. It goes without saying that any errors in the following notes are entirely my responsibility.

Part I

Nonlinear Wave Equations

2 Linear Wave Equation

2.1 Introduction

Through these notes, our goal is to study the properties of the nonlinear wave equation, which can either be a "simple" scalar equation or a system of equations [4]:

$$\frac{1}{\sqrt{-\det(g)}}\partial_\mu\left(\sqrt{-\det g}\,g^{\mu\nu}\partial_\nu\phi\right) = F(\phi,\partial\phi), \tag{1}$$

where g is a Lorentzian metric on $I \times \mathbb{R}^n$, and $\phi : I \times \mathbb{R}^n \to \mathbb{R}$ is an unknown function of some regularity. For example, if $n = 1$, and g is the flat Minkowski metric, $\mathrm{diag}(1,-1)$, then the nonlinear wave equation is merely

$$\partial_\mu(\partial^{\mu\nu}\partial_\nu\phi) = F(\phi,\partial\phi) \iff \partial_t^2 - (\partial_x\phi)^2 = F(\phi,\partial\phi).$$

Note that we denote the coordinate of I by t (to designate the *time* coordinate), and the coordinates of $\mathbb{R}^n$ by $x = (x^1,\dots,x^n)$. An equation of the form (1) is said to be

(i) **linear**: g is independent of ϕ and F is a linear function of both of its arguments;
(ii) **semilinear**: g is independent of ϕ and F is a nonlinear function;
(iii) **quasilinear**: g is independent of ϕ and/or $\partial\phi$.

Before we build up to the nonlinear wave equation, we will first examine the properties of the linear wave equation. Generally speaking, we are interested in the following properties:

(i) **Existence and Uniqueness**: Does a solution to (1) exist, and if so, is it the only solution?
(ii) **Dispersion**: What physical properties do solutions to (1) have?
(iii) **Well-Posedness**: If we consider the *Cauchy wave problem* consisting of the equation (1) along with initial data $(\phi,\partial_t\phi)$ on the hypersurface $\{t = 0\}$, where $\phi,\partial_t\phi$, does there exist a unique solution to the problem that satisfy the initial conditions and is continuously dependent on the initial data?

We will gradually work through each of these results, sometimes developing new techniques and machinery. We first begin by developing some general analytic results, and idea of energy estimates. Then we will apply these to the study of linear wave equations. This will then give a foundation for the study of nonlinear wave equations.

2.2 General Analytic Results

2.3 Energy Estimates

In this section, we introduce the notion of the energy of a solution to the wave equation. We begin by considering the homogeneous linear Cauchy problem,

$$\begin{cases} \square \phi = 0, \\ (\phi, \partial_t \phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1), \end{cases} \tag{2}$$

where we initially assume $\phi_0, \phi_1 \in C_c^\infty \subset \mathcal{S}(\mathbb{R}^n)$, and $\square$ denotes the **wave operator**. As of now, we do not yet have an existence (or uniqueness) result for this Cauchy wave problem. Thus we simply assume that such a unique (sufficiently regular) solution exists for now. Then we define the **energy** of a solution to be the following L^2-type norm,

Definition 2.1 (Energy)**.** *For $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n) \times C_c(\mathbb{R}^n)$ and ϕ the solution to the Cauchy problem (2), the* ***energy*** *of ϕ at the spacetime point (t, x) is defined to be the L^2-type norm,*

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left((\partial_t \phi)^2 + \sum_1^n (\partial_j \phi)^2 \right) (t, x) \, dx. \tag{3}$$

Intuitively, the definition for the energy is as we expected: consider an oscillating 1-dimensional string, and let $\phi(t, x)$ denote the amplitude of the wave at a point x at time t. Then the quantity $(1/2)(\partial_t \phi)^2$ resembles the kinetic energy of an infinitesimal point particle at the spacetime location (t, x), while the quantity $\sum_1^n (\partial_j \phi)^2$ resembles the (spring) potential energy at (t, x). Thus under these observations, $E(t)$ resembles the total energy of the system at some time t. Since the Cauchy wave problem (2) has no source term (i.e., is homogeneous), one expects the following result [4]:

Proposition 2.1 (Conservation of Energy)**.** *Let ϕ be a (sufficiently regular) solution to (2). Define $E(t)$ as given in Definition 2.1. Then E is constant in time.*

Remark 2.1. *We will give two proofs for this result; one relies on straightforward differentiation of $E(t)$ and follows the steps outlined in [4], while the other introduces the concept of conservation laws and follows the steps outlined in [3].*

Proof # 1. Let ϕ be a sufficiently regular solution to (2), and define the energy $E(t)$ of ϕ to be the function,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left((\partial_t \phi)^2 + \sum_1^n (\partial_j \phi)^2 \right) (t, x) \, dx,$$

where we assume that the initial data ϕ_0, ϕ_1 are both smooth with compact support. As we will see in a later section, $\phi \in C^\infty(I, \mathcal{S}(\mathbb{R}^n))$. Thus by the Dominated Convergence Theorem allows us to pass a time derivative inside the integral sign. We then obtain,

$$\begin{aligned}
\frac{d}{dt} E(t) &= \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t \phi)(\partial_t^2 \phi) + \sum_1^n \partial_t (\partial_j \phi)^2 \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t \phi) \cdot \left(\sum_1^n \partial_j^2 \phi \right) + \sum_1^n \partial_t \phi \cdot (\partial_j \phi)^2 \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t \phi) \cdot \left(\sum_1^n \partial_j^2 \phi \right) + \frac{1}{2} \sum_1^n \int_{\mathbb{R}^n} \partial_j (\partial_t \phi)(\partial_j \phi) \, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t \phi) \cdot \left(\sum_1^n \partial_j^2 \phi \right) + \frac{1}{2} \sum_1^n \left[(\partial_t \phi)(\partial_j \phi) \Big|_{|x| = \infty} - \int_{\mathbb{R}^n} (\partial_t \phi)(\partial_j^2 \phi) \, dx \right] \\
&= \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t \phi) \cdot \left(\sum_1^n \partial_j^2 \phi \right) dx - \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t \phi) \cdot \left(\sum_1^n \partial_j^2 \phi \right) dx = 0,
\end{aligned}$$

where the boundary term is identically zero since ϕ is Schwartz. Thus, the energy is conserved. $\square$

Proof # 2. We start by claiming that solutions to (2) are the stationary points of a variational integral. Define the linear functional $\mathcal{L} : C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n)) \to \mathbb{R}$ by

$$\mathcal{L}(\phi) = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}^n} (\partial_t \phi)^2 - \sum_1^n (\partial_j \phi)^2 \, dtdx;$$

to obtain this functional, we can use the Euler-Lagrange equations for a physical system. Now let $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$. We recall that $\langle \mathcal{L}'(\phi), \psi \rangle$, where $\mathcal{L}'(\phi)$ is the Fréchet derivative is defined as the directional derivative along ψ; i.e.,

$$\begin{aligned}
\langle \mathcal{L}'(\phi), \psi \rangle &= \frac{d}{d\varepsilon} \mathcal{L}(\phi + \varepsilon\psi)\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(\mathcal{L}(\phi) + \varepsilon \int_{\mathbb{R}\times\mathbb{R}^n} (-\nabla\phi \cdot \nabla\psi + (\partial_t\phi)(\partial_t\psi)) \, dtdx + \mathcal{O}(\varepsilon^2) \right) \\
&= \int_{\mathbb{R}\times\mathbb{R}^n} -(\nabla\phi) \cdot (\nabla\psi) + (\partial_t\phi)(\partial_t\psi) \, dtdx.
\end{aligned}$$

Since ψ is smooth with compact support, and ϕ is Schwartz, using Fubini's theorem and integration by parts, and recognizing that the boundary terms are identically zero, one obtains

$$\langle \mathcal{L}'(\phi), \psi \rangle = -\int_{\mathbb{R}\times\mathbb{R}^n} (\Box\phi)\psi \, dtdx = 0.$$

Thus since $\phi \in C_c^\infty$ was arbitrary, we conclude that ϕ is stationary for $\mathcal{L}$ iff $\Box\phi = 0$. Now, we recall Nöther's theorem, which states that if $\mathcal{L}$ is invariant with respect to a 1-parameter family of diffeomorphisms, then there exists a conservation law for an extreme value of $\mathcal{L}$. We note that $\mathcal{L}$ is time-translation invariant: if $G_\varepsilon : \phi(t,x) \mapsto \phi(t+\varepsilon, x)$, then one observes that $\mathcal{L}(\phi) = \mathcal{L}(G_\varepsilon\phi)$. Therefore, since the vector field ∂_t generates the flow corresponding to time translations, one has

$$\begin{aligned}
0 &= \phi_t \Box \phi \\
&= \phi_{tt}\phi_t - \Delta\phi\phi_t \\
&= \partial_t \left(\frac{(\partial_t\phi)^2}{2} - \frac{(\nabla\phi)^2}{2} \right) - \operatorname{div}(\nabla\phi\phi_t).
\end{aligned}$$

Then, integrating over $\mathbb{R}^n$ yields,

$$0 = \frac{d}{dt}\left(\frac{1}{2} \int_{\mathbb{R}^n} (\partial_t\phi)^2 - (\nabla\phi)^2 \, dx \right) - \int_{\mathbb{R}^n} \operatorname{div}(\nabla\phi\phi_t) \, dx.$$

The last term vanishes by using integration by parts and recalling that ϕ is Schwartz. Thus, one has that the energy is independent of time. $\square$

Remark 2.2. *As the second proof shows, because of the time translation invariance, to show that the energy is conserved, it suffices to integrate the quantity* $(\partial_t\phi)(\Box\phi)$ *by parts.*

Likewise, we have the following conservation law in Sobolev norms [2].

Corollary 2.1 (Sobolev Conservation Law I)**.** *Given initial data* $(\phi, \partial_t\phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$*, and assuming the existence of a solution* $\phi \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$ *to the Cauchy wave problem (2), for all* $s \in \mathbb{R}$*,*

$$\|\partial_t\phi\|_{H^{s-1}(\mathbb{R}^n)} + \|\nabla\phi\|_{H^{s-1}(\mathbb{R}^n)} = \|\phi_0\|_{H^{s-1}(\mathbb{R}^n)} + \|\nabla\phi_1\|_{H^{s-1}(\mathbb{R}^n)}. \tag{4}$$

Proof. Assume that $\phi \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$ is a solution to (2) with initial data $(\phi, \partial_t\phi) \restriction_{\{t=0\}} = (\phi_0, \phi_1) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$. If $\hat{\cdot}$ denotes the Fourier transform of $\cdot$, then one has the equation

$$(\partial_t\phi)^2 + 4\pi^2|\xi|^2(\hat{\phi})^2 = (\hat{\phi}_0)^2 + 4\pi^2|\xi|^2(\hat{\phi}_1)^2.$$

Note, this equation becomes clearer once we establish a representation formula for ϕ in a later section. Multiplying through by $(1+|\xi|^2)^{s-1}$ (for arbitrary $s \in \mathbb{R}$), and integrating, yields the desired conservation law. $\square$

One immediate advantage of having conservation laws is that if the Cauchy problem (2) has a solution, then these laws immediately imply uniqueness.

Corollary 2.2 (Uniqueness I)**.** *Given initial data* $(\phi, \partial_t \phi) \restriction|_{\{t=0\}} = (\phi_0, \phi_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$*, if we have two solutions* $\phi^{(1)}$ *and* $\phi^{(2)}$ *to the linear wave equation (2) with the given data such that both of them satisfy* $(\phi^{(i)}, \partial_t \phi^{(i)}) \in L^\infty([0,T]; H^1(\mathbb{R}^n)) \times L^\infty([0,T]; L^2(\mathbb{R}^n))$ *for* $i = 1, 2$*, then* $\phi^{(1)} = \phi^{(2)}$ *almost everywhere in* $[0,T] \times \mathbb{R}^n$*.*

Give proof (possibly using mollifiers). See Holzegel

A Explicit Representation Formulas

In this section, we work through the derivations of explicit representation formulas for initial value problems of the *wave equation*,

$$\Box u \coloneqq u_{tt} - \Delta u = 0 \quad \text{(or more generally, } f\text{)}, \tag{5}$$

where the unknown is the function $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x,t)$, with $x \in U^{\text{open}} \subseteq \mathbb{R}^n$ and $t > 0$, Δ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$, and $f : U \times [0, \infty) \to \mathbb{R}$ is a given function [1].

A.0.1 The Wave Equation in One Spatial Dimension – D'Alembert's Formula

The simplest example of the wave equation happens in the case $n = 1$. Consider the following initial value problem,

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{6}$$

We note that the wave equation can be written in a more suggestive manner:

$$0 = u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)u \eqqcolon (\partial_t + \partial_x)v, \tag{7}$$

where we set $v = v(x,t) = u_t - u_x$. Thus, v satisfies the transport equation, $v_t + v_x = 0$. Using the solution to the transport equation in one dimension, one obtains the solution,

$$v(x,t) = a(x-t),$$

where $a(x) \coloneqq v(x,0)$. This gives us the nonhomogeneous transport equation,

$$u_t - u_x = a(x-t),$$

for which the explicit representation formula was shown (see Appendix) to be

$$u(x,t) = \int_0^t a(x + (t-s) - s)\, ds + g(x+t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)\, dy + g(x+t). \tag{8}$$

It remains for us to identify the function $a(x,t)$. Using the definition of $v(x,t)$, at $t = 0$, one has,

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x).$$

Therefore, using the fundamental theorem of calculus and the above results, we conclude that

$$\boxed{u(x,t) = \frac{1}{2}\left[g(x+t) - g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy} \qquad (x \in \mathbb{R}^n, t \geq 0). \tag{9}$$

This explicit formula is called **d'Alembert's formula** for $n = 1$. Hence, we have shown that *if* the wave equation has a "sufficiently smooth solution u", then it must be given by d'Alembert's formula. The following theorem formalizes this result.

Theorem A.1 (Solution to the Wave Equation ($n = 1$)). *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and define u by d'Alembert's formula. Then*

(i) $u \in C^2(\mathbb{R} \times [0, \infty))$.

(ii) $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times (0, \infty)$

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ t>0}} u(x,t) = g(x^0)$ *and* $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ t>0}} u_t(x,t) = h(x^0)$ *for each point* $x^0 \in \mathbb{R}$.

Proof.

(i) Define u by d'Alembert's formula. Then we compute the following:

$$u_x = \frac{1}{2}[g'(x+t) + g'(x-t)] + \frac{1}{2}[h(x+t) - h(x-t)].$$
$$u_t = \frac{1}{2}[g'(x+t) - g'(x-t)] + \frac{1}{2}[h(x+t) + h(x-t)],$$

whence

$$u_{xx} = \frac{1}{2}[g''(x+t) + g''(x-t)] + \frac{1}{2}[h'(x+t) - h'(x-t)].$$
$$u_{xt} = \frac{1}{2}[g''(x+t) - g''(x-t)] + \frac{1}{2}[h'(x+t) + h'(x-t)].$$
$$u_{tt} = \frac{1}{2}[g''(x+t) + g''(x-t)] + \frac{1}{2}[h'(x+t) - h'(x-t)].$$

Since g', g'' and h, h' are all continuous at every $x \in \mathbb{R}$ and $t \in [0, \infty)$, as are the functions $x + t$ and $x - t$, we conclude that each of the first and second-order partial derivatives of u are continuous on $\mathbb{R} \times [0, \infty)$. Thus, $u \in C^2(\mathbb{R} \times [0, \infty))$.

(ii) From the above calculations, it follows immediately that $u_{tt} - u_{xx} = 0$.

(iii) By (1), u is continuous at every $(x, t) \in \mathbb{R} \times [0, \infty)$, and thus continuous at every point of the form $(x^0, 0) \in \mathbb{R} \times \{0\}$. Thus by the limit definition of continuity, one has

$$\lim_{(x,t)\to(x^0,0)} u(x,t) = u(x^0, 0) =: g(x^0).$$

Likewise, by our remark above, $u_t(x,t) \in C^1(\mathbb{R} \times [0, \infty)$. Thus by the limit definition of continuity, one has

$$\lim_{(x,t)\to(x^0,0)} u_t(x,t) = u_t(x^0, 0) =: h(x^0).$$

□

A.0.2 The Wave Equation in One Spatial Dimension – Half-Line

Now we will consider the modified initial value problem,

$$\begin{cases} u_{tt} - u_{xx} = 0 & (\text{in } \mathbb{R}_+ \times (0, \infty) \\ u = g, u_t = h & (\text{in } \mathbb{R}_+ \times \{t = 0\}) \\ u = 0 & (\text{on } \{x = 0\} \times (0, \infty). \end{cases} \tag{10}$$

To solve this problem, we will use the *method of reflections*; i.e., we will extend the initial value problem to the entire real line by means of the odd reflections:

$$\widetilde{u}(x,t) = \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x \leq 0. \end{cases}$$

$$\widetilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x \leq 0, \end{cases}$$

$$\widetilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x \leq 0. \end{cases}$$

It is straightforward to check that $\widetilde{u}, \widetilde{g}, \widetilde{h}$ satisfy the ordinary wave equation ($n = 1$) on the whole real line. Thus by d'Alembert's formula, we obtain the following solution:

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) - g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & (x \geq t \geq 0) \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy & (0 \leq x \leq t) \end{cases}} \tag{11}$$

A.0.3 The Wave Equation in $n \geq 2$ Spatial Dimension – Kirchoff's and Poisson's Formulas

In this subsection, we will develop explicit representation formulas for the wave equation for dimensions $n \geq 2$. To develop these formulas, we require the *method of spherical means.* Consider the following definitions.

Definition A.1 (Spherical Means)**.**

(i) Let $x \in \mathbb{R}^n$, $t > 0$, and $r > 0$. We define the average of $u(\cdot,t)$ over the sphere $\partial B(x,r)$ to be

$$U(x;r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS(y). \tag{12}$$

(ii) We make the analogous definitions,

$$\begin{cases} G(x;r) &= \fint_{\partial B(x,r)} g(y)\,dS(y). \\ H(x;r) &= \fint_{\partial B(x,r)} h(y)\,dS(y). \end{cases} \tag{13}$$

For any fixed x, we will regard U, G, H as functions of r and t.

Lemma A.1 (Euler-Poisson-Darboux Equation)**.** *Fix $x \in \mathbb{R}^n$, $n, m \geq 2$, and let $u \in C^m(\mathbb{R}^n \times [0,\infty))$ satisfy the initial value problem,*

$$\begin{cases} u_{tt} - \Delta u = 0 & (in\ \mathbb{R}^n \times (0,\infty)) \\ u = g, u_t = h & (in\ \mathbb{R}^n \times \{t = 0\}). \end{cases}$$

Then $U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty))$, and U_{tt} satisfies the Euler-Poisson-Darboux initial value problem,

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & (in\ \mathbb{R}_+ \times (0,\infty)) \\ U = G, U_t = H & (on\ \mathbb{R}_+ \times \{t = 0\} \end{cases} \tag{14}$$

Proof. We begin by taking the partial derivative of the spherical mean U with respect to r. To do so, we rewrite

$$U(x;r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS(y) = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} u(y,t)\,dS(y) = \frac{1}{n\alpha(n)}\int_{\partial B(0,1)} u(x+rz,t)\,dS(z).$$

Thus,

$$\begin{aligned} U_r(x;r,t) &= \frac{1}{n\alpha(n)}\frac{\partial}{\partial r}\int_{\partial B(0,1)} u(x+rz,t)\,dS(z) \\ &= \frac{1}{n\alpha(n)}\int_{\partial B(0,1)} Du(x+rz,t)\cdot z\,dS(z) \\ &= \frac{r^{1-n}}{n\alpha(n)}\int_{\partial B(x,r)} Du(y,t)\cdot\frac{y-x}{r}\,dS(y) \\ &= \frac{r}{n\alpha(n)r^n}\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\,dS(y) \\ &= \frac{r}{n\alpha(n)r^n}\int_{B(x,r)} \Delta u\,dy = \frac{r}{n}\fint_{B(x,r)} \Delta u\,dy, \end{aligned}$$

where passing the derivative inside the integral in the second line is justified since u is sufficiently smooth (meaning that Du exists a.e. and is integrable on $\partial B(0,1)$), and the final line follows from Green's formula. Now taking some large $R > 0$, and restricting to $0 < r \le R/2$, one has that since $B(x,r) \subset \overline{B(x,R/2)}$ and Δu is continuous (so that the extreme value theorem applies),

$$\|\Delta u\|_{L^\infty(B(x,r))} \le \|\Delta u\|_{L^\infty(B(x,R))} =: C < \infty.$$

Hence,

$$U_r(x;r,t) \le \frac{C}{n} r \implies \lim_{r\to 0^+} U_r(x;r,t) = 0.$$

Now we will compute the second partial derivative of U with respect to r. We compute,

$$\begin{aligned}
U_{rr}(x;r,t) &= \frac{1-n}{n\alpha(n)r^n} \int_{B(x,r)} \Delta u \, dy + \frac{r^{1-n}}{n\alpha(n)} \frac{\partial}{\partial r} \int_{B(x,r)} \Delta u(y,t)\, dy \\
&= \frac{1-n}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy + \frac{r^{1-n}}{n\alpha(n)} \int_{\partial B(x,r)} \Delta u(y,t)\, dS(y) \\
&= \fint_{\partial B(x,r)} \Delta u(y,t)\, dS(y) + \left(\frac{1}{n} - 1\right) \fint_{B(x,r)} \Delta u(y,t)\, dy,
\end{aligned}$$

where the second line follows from the coarea formula (see Appendix). By applying similar reasoning to above, one computes that $\lim_{r\to 0^+} U_{rr}(x;r,t) = (1/n)\Delta u$. In this manner, repeatedly differentiating U, one sees that $U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty))$. Now differentiating,

$$\begin{aligned}
(n-1)r^{n-2}U_r + r^{n-1}U_{rr} &= \frac{d}{dr}(r^{n-1}U_r) \\
&= \frac{d}{dr} \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}\, dy \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}\, dS(y) \\
&= r^{n-1} \fint_{\partial B(x,r)} u_{tt}\, dS(y) =: r^{n-1}U_{tt},
\end{aligned}$$

with the third line following from the coarea formula. □

Now we will derive the explicit representation formula for the 3-dimensional wave equation by reducing the system to the one-dimensional Euler-Poisson-Darboux equation and solving this system using the formula we derived for the one-dimensional half-line equation.

Kirchoff's Formula for the Wave Equation ($n = 3$) Suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & (\text{in } \mathbb{R}^3 \times [0,\infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R}^3 \times \{t = 0\}) \end{cases} \tag{15}$$

Define the functions, $\widetilde{U} = rU(x;r,t)$, $\widetilde{G}(r) = rG(x;r)$, and $\widetilde{H}(r) = rH(x;r)$. Then we observe the following:

(i)
$$\begin{aligned}
\widetilde{U}_{tt} = rU_{tt} &= r\left[U_{rr} + \frac{2}{r}U_r\right] \\
&= rU_{rr} + 2U_r = (U + rU_r)_r = \widetilde{U}_{rr}.
\end{aligned}$$

(ii) On $\mathbb{R}_+ \times \{t = 0\}$, $U = G$ and $U_t = H$ so that $\widetilde{U} = \widetilde{G}$ and $\widetilde{U}_t = \widetilde{H}$.

(iii) For $r = 0$, $\partial B(x,0) = \varnothing$ so that $\widetilde{U} = 0$ on $\{r = 0\} \times (0,\infty)$.

In other words, we have shown that $\widetilde{U}$ satisfies the following initial value problem,

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} \ = 0 & (\text{in } \mathbb{R}_+ \times (0,\infty)) \\ \widetilde{U} = \widetilde{G}, \widetilde{U}_t = \widetilde{H} & (\text{on } \mathbb{R}_+ \times \{t=0\}) \\ \widetilde{U} = 0 & (\text{on } \{r=0\} \times (0,\infty)) \end{cases}$$

Using the solution for the one-dimensional wave equation on the half-line, for $0 \le r \le t$, one has

$$\widetilde{U}(x;r,t) = \frac{1}{2}[\widetilde{G}(r+t) - \widetilde{G}(t-r)] + \frac{1}{2}\int_{t-r}^{r+t} \widetilde{H}(y)\, dy.$$

On the other hand, one has that

$$\begin{aligned} u(x,t) &= \lim_{r\to 0^+} \frac{\widetilde{U}(x;r,t)}{r} \\ &= \lim_{r\to 0^+} \left[\frac{\widetilde{G}(r+t) - \widetilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{r+t} \widetilde{H}(y)\, dy \right] \\ &= \widetilde{G}'(t) + \widetilde{H}(t). \end{aligned}$$

Recall again that

$$\begin{aligned} \widetilde{G}'(t) &= \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g(y)\, dS(y) \right) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{n\alpha(n)t} \int_{\partial B(x,t)} g(y)\, dS(y) \right) \\ &= -\frac{1}{n\alpha(n)t^2}\int_{\partial B(x,t)} g(y)\, dS(y) + \frac{1}{n\alpha(n)t}\frac{\partial}{\partial t}\left(t^2 \int_{\partial B(0,1)} g(x+tz)\, dS(z) \right) \\ &= -\fint_{\partial B(x,t)} g(y)\, dS(y) + \frac{2}{n\alpha(n)}\int_{\partial B(0,1)} g(x+tz)\, dS(z) + \frac{1}{n\alpha(n)t}\int_{\partial B(x,t)} Dg(y)\cdot\left(\frac{y-x}{t}\right) dS(y) \\ &= \fint_{\partial B(x,t)} g(y)\, dS(y) + \fint_{\partial B(x,t)} Dg(y)\cdot(y-x)\, dS(y). \end{aligned} \tag{16}$$

Likewise, one has

$$\widetilde{H}(t) = tH(t) = \fint_{\partial B(x,t)} th(y)\, dS(y).$$

Therefore, we conclude that

$$\boxed{u(x,t) = \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\, dS(y)} \tag{17}$$

This is **Kirchoff's formula** for the wave equation in dimension 3.

Poisson's Formula for the Wave Equation ($n=2$) Assume $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves the IVP for $n=2$. Unlike for the $n=3$ case, we will consider the 2-dimensional problem and "view" it as a 3-dimensional problem. Define $\bar{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$ so that u satisfies the IVP

$$\begin{cases} \bar{u}_{tt} - \Delta\bar{u} = 0 & (\text{in } \mathbb{R}^3 \times (0,\infty)) \\ \bar{u} = \bar{g}, \bar{u}_t = \bar{h} & (\text{on } \mathbb{R}^3 \times \{t=0\}), \end{cases}$$

where we define $\bar{g}$ and $\bar{h}$ analogously. For $x = (x_1,x_2) \in \mathbb{R}^2$, write $\bar{x} = (x_1,x_2,0) \in \mathbb{R}^3$. Then by Kirchoff's formula, one has

$$u(x,t) = \bar{u}(\bar{x},t) = \frac{\partial}{\partial t}\left(t\fint_{\partial\bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} \right) + t\fint_{\partial\bar{B}(\bar{x},t)} \bar{h}\, d\bar{S},$$

where $\bar{B}(\bar{x},t)$ denotes the ball in $\mathbb{R}^3$ with center $\bar{x}$, radius $t>0$, and where $d\bar{S}$ denotes the 2-dimensional surface measure on $\partial\bar{B}(\bar{x},t)$. But we can write

$$\fint_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{S} = \frac{1}{4\pi t^2}\int_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{S} = \frac{2}{4\pi t^2}\int_{B(x,t)} g(y)(1+|D\gamma(y)|^2)^{1/2}\,dy,$$

where $\gamma(y) = (t^2-|y-x|^2)^{1/2}$ for $y\in B(x,t)$. The factor 2 comes from the two hemispheres of $\partial\bar{B}(\bar{x},t)$. Since $(1+|D\gamma|^2)^{1/2} = t(t^2-|y-x|^2)^{-1/2}$, one has

$$\begin{aligned}\fint_{\partial\bar{B}(\bar{x},t)} \bar{g}d\bar{S} &= \frac{1}{2\pi t}\int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\\ &= \frac{t}{2}\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy.\end{aligned}$$

Using this and the coarea formula, one obtains

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) + \frac{t^2}{2}\fint_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy.$$

Using the same reasoning as above to pass the derivative inside the integral, one has

$$\frac{\partial}{\partial t}\left(t^2\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) = t\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy + t\fint_{B(x,t)} \frac{Dg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy.$$

Thus for $n=2$, we obtain **Poisson's Formula**,

$$\boxed{u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{tg(y)+t^2h(y)+tDg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy \qquad (x\in\mathbb{R}^2, t>0)} \tag{18}$$

Notice that in solving the wave equation for $n=2$, we used the *method of descent*; i.e., we solved the problem for $n=3$ first, and then dropped down to $n=2$.

General Solutions for $n\geq 2$ Following similar extensive calculations as above (see Appendix for the work), we obtain the following solutions to the wave equation,

$$\boxed{u(x,t) = \begin{cases} \dfrac{1}{\gamma(n)}\left[\left(\dfrac{\partial}{\partial t}\right)\left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} g\,dS\right) + \left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} h\,dS\right)\right] & \begin{matrix} n\geq 3\\ n \text{ odd}\end{matrix}\\ \dfrac{1}{\beta(n)}\left[\left(\dfrac{\partial}{\partial t}\right)\left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n}\fint_{B(x,t)} \dfrac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\right. & \\ \qquad\left. + \left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n}\fint_{B(x,t)} \dfrac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\right] & \begin{matrix} n\geq 2\\ n \text{ even}\end{matrix}\end{cases}} \tag{19}$$

where in the first equation $\gamma_n = 1\cdot 3\cdot\cdots\cdot(n-2)$, and $\beta(n) = 2\cdot 4\cdot\cdots\cdot(n-2)\cdot n$. The above solutions highlight an interesting contrast between odd-dimensional, and even-dimensional solutions. Note that if n is odd and $n\geq 3$, the initial data at a point $x\in\mathbb{R}^n$ only affect the solution u along the boundary $\{(y,t)\mid t>0, |x-y|=t\}$ of the cone $C=\{(y,t)\mid t>0, |x-y|<t\}$. On the other hand, if n is given, the initial data can affect the solution within all of C. Thus, a disturbance originating at x propagates along a sharp wavefront in odd dimensions, but continues to have effects even after the leading edge of the wavefront passes in even dimensions. This phenomenon is **Huygen's principle**.

A.1 Solution to the Nonhomogeneous Wave Equation

Now we will use the solutions from the previous sections to address the nonhomogeneous wave equation,

$$\begin{cases} u_{tt} - \Delta u = f & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = 0, u_t = 0 & (\text{on } \mathbb{R}^n \times \{t = 0\}). \end{cases} \tag{20}$$

To solve this problem, we use Duhamel's principle[1] by setting $u = u(x,t;s)$ to be the solution of

$$\begin{cases} u_{tt}(\cdot;s) - \Delta u(\cdot,s) & = 0 & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u(\cdot;s) = 0, u_t(\cdot;s) = f(\cdot,s) & & (\text{on } \mathbb{R}^n \times \{t = 0\}). \end{cases} \tag{21}$$

Thus, by Duhamel's principle,

$$u(x,t) \coloneqq \int_0^t u(x,t;s)\, ds \qquad (x \in \mathbb{R}^n, t \geq 0) \tag{22}$$

is a solution of the above nonhomogeneous wave equation. Consequently, the solution to the general inhomogeneous wave problem,

$$\begin{cases} u_{tt} - \Delta u = f & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R}^n \times \{t = 0\}) \end{cases} \tag{23}$$

is equal to the sum of the solutions to the problems

$$\begin{cases} u_{tt} - \Delta u = 0 & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R}^n \times \{t = 0\}) \end{cases} \quad \text{and} \quad \begin{cases} u_{tt} - \Delta u = f & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = 0, u_t = 0 & (\text{on } \mathbb{R}^n \times \{t = 0\}) \end{cases} \tag{24}$$

A.1.1 Example 1: Solution to Nonhomogeneous Problem for $n = 1$

By d'Alembert's formula, one has

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t+s} f(y,s)\, dy.$$

Thus,

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t+s} f(y,s)\, dyds = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\, dyds \qquad (x \in \mathbb{R}, t \geq 0).$$

A.1.2 Example 2: Solution to Nonhomogeneous Problem for $n = 3$

By Kirchoff's formula, one has

$$u(x,t;s) = (t-s)\fint_{\partial B(x,t-s)} f(y,s)\, dS.$$

This implies that

$$\begin{aligned} u(x,t) &= \int_0^t (t-s)\fint_{\partial B(x,t-s)} f(y,s)\, dSds \\ &= \int_0^t (t-s)\frac{1}{4\pi(t-s)^2}\int_{\partial B(x,t-s)} f(y,s)\, dSds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\, dSds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dSdr. \end{aligned}$$

[1]See https://en.wikipedia.org/wiki/Duhamel%27s_principle for an explanation.

Hence, we conclude that

$$u(x,t) = \frac{1}{4\pi} \int_{B(x,t)} \frac{f(y, t - |y - x|)}{|y - x|} \, dy \qquad (x \in \mathbb{R}^3, t \geq 0),$$

where the integrand on the right is called a *retarded potential*.

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