Hyperbolic Partial Differential Equations

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June 2026

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Part I
The Wave Equation

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.
>
> John Scott Russell

Resources:

1. *Partial Differential Equations* (2nd Ed.), Lawrence C. Evans
2. *Introduction to Nonlinear Wave Equations*, Jonathan Luk
3. *Nonlinear Wave Equations*, Gustav Holzegel

1 Introduction to the Wave Equation

1.1 Explicit Representation Formulas

In this section, we talk will work through the derivations of explicit representation formulas for initial value problems of the *wave equation*,

$$\Box u \coloneqq u_{tt} - \Delta u = 0 \quad \text{(or more generally, } f\text{)}, \tag{1.1}$$

where the unknown is the function $u : \bar{U} \times [0, \infty) \to \mathbb{R}$, $u = u(x, t)$, with $x \in U^{\text{open}} \subseteq \mathbb{R}^n$ and $t > 0$, Δ is taken with respect to the spatial variables $x = (x_1, \dots, x_n)$, and $f : U \times [0, \infty) \to \mathbb{R}$ is a given function.

1.1.1 The Wave Equation in One Spatial Dimension – D'Alembert's Formula

The simplest example of the wave equation happens in the case $n = 1$. Consider the following initial value problem,

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{1.2}$$

We note that the wave equation can be written in a more suggestive manner:

$$0 = u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)u \eqqcolon (\partial_t + \partial_x)v, \tag{1.3}$$

where we set $v = v(x, t) = u_t - u_x$. Thus, v satisfies the transport equation, $v_t + v_x = 0$. Using the solution to the transport equation in one dimension, one obtains the solution,

$$v(x, t) = a(x - t),$$

where $a(x) \coloneqq v(x, 0)$. This gives us the nonhomogeneous transport equation,

$$u_t - u_x = a(x - t),$$

for which the explicit representation formula was shown (see Appendix) to be

$$u(x, t) = \int_0^t a(x + (t - s) - s)\, ds + g(x + t) = \frac{1}{2} \int_{x-t}^{x+t} a(y)\, dy + g(x + t). \tag{1.4}$$

It remains for us to identify the function $a(x, t)$. Using the definition of $v(x, t)$, at $t = 0$, one has,

$$a(x) = v(x, 0) = u_t(x, 0) - u_x(x, 0) = h(x) - g'(x).$$

Therefore, using the fundamental theorem of calculus and the above results, we conclude that

$$\boxed{u(x, t) = \frac{1}{2}[g(x + t) - g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy} \qquad (x \in \mathbb{R}^n, t \geq 0). \tag{1.5}$$

This explicit formula is called **d'Alembert's formula** for $n = 1$. Hence, we have shown that *if* the wave equation has a "sufficiently smooth solution u", then it must be given by d'Alembert's formula. The following theorem formalizes this result.

Theorem 1.1 (Solution to the Wave Equation ($n = 1$))**.** *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and define u by d'Alembert's formula. Then*

(i) $u \in C^2(\mathbb{R} \times [0, \infty))$.

(ii) $u_{tt} - u_{xx} = 0$ *in* $\mathbb{R} \times (0, \infty)$

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ t>0}} u(x, t) = g(x^0)$ *and* $\displaystyle\lim_{\substack{(x,t)\to(x^0,0) \\ t>0}} u_t(x, t) = h(x^0)$ *for each point* $x^0 \in \mathbb{R}$.

Proof.

(i) Define u by d'Alembert's formula. Then we compute the following:

$$u_x = \frac{1}{2}[g'(x+t) + g'(x-t)] + \frac{1}{2}[h(x+t) - h(x-t)].$$
$$u_t = \frac{1}{2}[g'(x+t) - g'(x-t)] + \frac{1}{2}[h(x+t) + h(x-t)],$$

whence

$$u_{xx} = \frac{1}{2}[g''(x+t) + g''(x-t)] + \frac{1}{2}[h'(x+t) - h'(x-t)].$$
$$u_{xt} = \frac{1}{2}[g''(x+t) - g''(x-t)] + \frac{1}{2}[h'(x+t) + h'(x-t)].$$
$$u_{tt} = \frac{1}{2}[g''(x+t) + g''(x-t)] + \frac{1}{2}[h'(x+t) - h'(x-t)].$$

Since g', g'' and h, h' are all continuous at every $x \in \mathbb{R}$ and $t \in [0,\infty)$, as are the functions $x+t$ and $x-t$, we conclude that each of the first and second-order partial derivatives of u are continuous on $\mathbb{R} \times [0,\infty)$. Thus, $u \in C^2(\mathbb{R} \times [0,\infty))$.

(ii) From the above calculations, it follows immediately that $u_{tt} - u_{xx} = 0$.

(iii) By (1), u is continuous at every $(x,t) \in \mathbb{R} \times [0,\infty)$, and thus continuous at every point of the form $(x^0, 0) \in \mathbb{R} \times \{0\}$. Thus by the limit definition of continuity, one has

$$\lim_{(x,t)\to(x^0,0)} u(x,t) = u(x^0,0) =: g(x^0).$$

Likewise, by our remark above, $u_t(x,t) \in C^1(\mathbb{R} \times [0,\infty)$. Thus by the limit definition of continuity, one has

$$\lim_{(x,t)\to(x^0,0)} u_t(x,t) = u_t(x^0,0) =: h(x^0).$$

□

1.1.2 The Wave Equation in One Spatial Dimension – Half-Line

Now we will consider the modified initial value problem,

$$\begin{cases} u_{tt} - u_{xx} = 0 & (\text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g, u_t = h & (\text{in } \mathbb{R}_+ \times \{t = 0\}) \\ u = 0 & (\text{on } \{x = 0\} \times (0,\infty). \end{cases} \tag{1.6}$$

To solve this problem, we will use the *method of reflections*; i.e., we will extend the initial value problem to the entire real line by means of the odd reflections:

$$\widetilde{u}(x,t) = \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x \leq 0. \end{cases}$$

$$\widetilde{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x \leq 0, \end{cases}$$

$$\widetilde{h}(x) = \begin{cases} h(x), & x \geq 0 \\ -h(-x), & x \leq 0. \end{cases}$$

It is straightforward to check that $\widetilde{u}, \widetilde{g}, \widetilde{h}$ satisfy the ordinary wave equation ($n = 1$) on the whole real line. Thus by d'Alembert's formula, we obtain the following solution:

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) - g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy & (x \geq t \geq 0) \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{t-x}^{x+t} h(y)\, dy & (0 \leq x \leq t) \end{cases}} \tag{1.7}$$

1.1.3 The Wave Equation in $n \geq 2$ Spatial Dimension – Kirchoff's and Poisson's Formulas

In this subsection, we will develop explicit representation formulas for the wave equation for dimensions $n \geq 2$. To develop these formulas, we require the *method of spherical means.* Consider the following definitions.

Definition 1.1 (Spherical Means)**.**

(i) Let $x \in \mathbb{R}^n$, $t > 0$, and $r > 0$. We define the average of $u(\cdot, t)$ over the sphere $\partial B(x,r)$ to be

$$U(x; r, t) = \fint_{\partial B(x,r)} u(y,t)\, dS(y). \tag{1.8}$$

(ii) We make the analogous definitions,

$$\begin{cases} G(x;r) &= \fint_{\partial B(x,r)} g(y)\, dS(y). \\ H(x;r) &= \fint_{\partial B(x,r)} h(y)\, dS(y). \end{cases} \tag{1.9}$$

For any fixed x, we will regard U, G, H as functions of r and t.

Lemma 1.1 (Euler-Poisson-Darboux Equation)**.** *Fix $x \in \mathbb{R}^n$, $n, m \geq 2$, and let $u \in C^m(\mathbb{R}^n \times [0, \infty))$ satisfy the initial value problem,*

$$\begin{cases} u_{tt} - \Delta u = 0 & (in\ \mathbb{R}^n \times (0, \infty)) \\ u = g, u_t = h & (in\ \mathbb{R}^n \times \{t = 0\}). \end{cases}$$

Then $U \in C^m(\overline{\mathbb{R}}_+ \times [0, \infty))$, and U_{tt} satisfies the Euler-Poisson-Darboux initial value problem,

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 & (in\ \mathbb{R}_+ \times (0, \infty)) \\ U = G, U_t = H & (on\ \mathbb{R}_+ \times \{t = 0\} \end{cases} \tag{1.10}$$

Proof. We begin by taking the partial derivative of the spherical mean U with respect to r. To do so, we rewrite

$$U(x; r, t) = \fint_{\partial B(x,r)} u(y,t)\, dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u(y,t)\, dS(y) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x + rz, t)\, dS(z).$$

Thus,

$$\begin{aligned} U_r(x; r, t) &= \frac{1}{n\alpha(n)} \frac{\partial}{\partial r} \int_{\partial B(0,1)} u(x + rz, t)\, dS(z) \\ &= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x + rz, t) \cdot z\, dS(z) \\ &= \frac{r^{1-n}}{n\alpha(n)} \int_{\partial B(x,r)} Du(y,t) \cdot \frac{y - x}{r}\, dS(y) \\ &= \frac{r}{n\alpha(n) r^n} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS(y) \\ &= \frac{r}{n\alpha(n) r^n} \int_{B(x,r)} \Delta u\, dy = \frac{r}{n} \fint_{B(x,r)} \Delta u\, dy, \end{aligned}$$

where passing the derivative inside the integral in the second line is justified since u is sufficiently smooth (meaning that Du exists a.e. and is integrable on $\partial B(0,1)$), and the final line follows from Green's formula. Now taking some large $R > 0$, and restricting to $0 < r \leq R/2$, one has that since $B(x,r) \subset \overline{B(x, R/2)}$ and Δu is continuous (so that the extreme value theorem applies),

$$\|\Delta u\|_{L^\infty(B(x,r))} \leq \|\Delta u\|_{L^\infty(B(x,R))} =: C < \infty.$$

Hence,

$$U_r(x;r,t) \leq \frac{C}{n} r \implies \lim_{r\to 0^+} U_r(x;r,t) = 0.$$

Now we will compute the second partial derivative of U with respect to r. We compute,

$$\begin{aligned} U_{rr}(x;r,t) &= \frac{1-n}{n\alpha(n) r^n} \int_{B(x,r)} \Delta u \, dy + \frac{r^{1-n}}{n\alpha(n)} \frac{\partial}{\partial r} \int_{B(x,r)} \Delta u(y,t)\, dy \\ &= \frac{1-n}{n} \fint_{B(x,r)} \Delta u(y,t)\, dy + \frac{r^{1-n}}{n\alpha(n)} \int_{\partial B(x,r)} \Delta u(y,t)\, dS(y) \\ &= \fint_{\partial B(x,r)} \Delta u(y,t)\, dS(y) + \left(\frac{1}{n} - 1\right) \fint_{B(x,r)} \Delta u(y,t)\, dy, \end{aligned}$$

where the second line follows from the coarea formula (see Appendix). By applying similar reasoning to above, one computes that $\lim_{r\to 0^+} U_{rr}(x;r,t) = (1/n)\Delta u$. In this manner, repeatedly differentiating U, one sees that $U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty))$. Now differentiating,

$$\begin{aligned} (n-1) r^{n-2} U_r + r^{n-1} U_{rr} &= \frac{d}{dr}(r^{n-1} U_r) \\ &= \frac{d}{dr} \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}\, dy \\ &= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}\, dS(y) \\ &= r^{n-1} \fint_{\partial B(x,r)} u_{tt}\, dS(y) =: r^{n-1} U_{tt}, \end{aligned}$$

with the third line following from the coarea formula. □

Now we will derive the explicit representation formula for the 3-dimensional wave equation by reducing the system to the one-dimensional Euler-Poisson-Darboux equation and solving this system using the formula we derived for the one-dimensional half-line equation.

Kirchoff's Formula for the Wave Equation ($n = 3$) Suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & (\text{in } \mathbb{R}^3 \times [0,\infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R}^3 \times \{t = 0\}) \end{cases} \tag{1.11}$$

Define the functions, $\widetilde{U} = rU(x;r,t)$, $\widetilde{G}(r) = rG(x;r)$, and $\widetilde{H}(r) = rH(x;r)$. Then we observe the following:

(i)

$$\begin{aligned} \widetilde{U}_{tt} = rU_{tt} &= r\left[U_{rr} + \frac{2}{r} U_r\right] \\ &= rU_{rr} + 2U_r = (U + rU_r)_r = \widetilde{U}_{rr}. \end{aligned}$$

(ii) On $\mathbb{R}_+ \times \{t = 0\}$, $U = G$ and $U_t = H$ so that $\widetilde{U} = \widetilde{G}$ and $\widetilde{U}_t = \widetilde{H}$.

(iii) For $r = 0$, $\partial B(x,0) = \varnothing$ so that $\widetilde{U} = 0$ on $\{r = 0\} \times (0,\infty)$.

In other words, we have shown that $\widetilde{U}$ satisfies the following initial value problem,

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 & (\text{in } \mathbb{R}_+ \times (0,\infty)) \\ \widetilde{U} = \widetilde{G}, \widetilde{U}_t = \widetilde{H} & (\text{on } \mathbb{R}_+ \times \{t = 0\}) \\ \widetilde{U} = 0 & (\text{on } \{r = 0\} \times (0,\infty)) \end{cases}$$

Using the solution for the one-dimensional wave equation on the half-line, for $0 \leq r \leq t$, one has

$$\widetilde{U}(x;r,t) = \frac{1}{2}[\widetilde{G}(r+t) - \widetilde{G}(t-r)] + \frac{1}{2} \int_{t-r}^{r+t} \widetilde{H}(y)\, dy.$$

On the other hand, one has that

$$\begin{aligned} u(x,t) &= \lim_{r\to 0^+} \frac{\widetilde{U}(x;r,t)}{r} \\ &= \lim_{r\to 0^+} \left[\frac{\widetilde{G}(r+t)-\widetilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{r+t} \widetilde{H}(y)\,dy \right] \\ &= \widetilde{G}'(t) + \widetilde{H}(t). \end{aligned}$$

Recall again that

$$\begin{aligned} \widetilde{G}'(t) &= \frac{\partial}{\partial t}\left(t \fint_{\partial B(x,t)} g(y)\,dS(y) \right) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{n\alpha(n)t}\int_{\partial B(x,t)} g(y)\,dS(y) \right) \\ &= -\frac{1}{n\alpha(n)t^2}\int_{\partial B(x,t)} g(y)\,dS(y) + \frac{1}{n\alpha(n)t}\frac{\partial}{\partial t}\left(t^2 \int_{\partial B(0,1)} g(x+tz)\,dS(z) \right) \\ &= -\fint_{\partial B(x,t)} g(y)\,dS(y) + \frac{2}{n\alpha(n)}\int_{\partial B(0,1)} g(x+tz)\,dS(z) + \frac{1}{n\alpha(n)t}\int_{\partial B(x,t)} Dg(y)\cdot\left(\frac{y-x}{t}\right)dS(y) \\ &= \fint_{\partial B(x,t)} g(y)\,dS(y) + \fint_{\partial B(x,t)} Dg(y)\cdot(y-x)\,dS(y). \end{aligned} \tag{1.12}$$

Likewise, one has

$$\widetilde{H}(t) = tH(t) = \fint_{\partial B(x,t)} th(y)\,dS(y).$$

Therefore, we conclude that

$$\boxed{u(x,t) = \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\,dS(y)} \tag{1.13}$$

This is **Kirchoff's formula** for the wave equation in dimension 3.

Poisson's Formula for the Wave Equation $(n=2)$ Assume $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves the IVP for $n=2$. Unlike for the $n=3$ case, we will consider the 2-dimensional problem and "view" it as a 3-dimensional problem. Define $\bar{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$ so that u satisfies the IVP

$$\begin{cases} \bar{u}_{tt} - \Delta\bar{u} = 0 & (\text{in } \mathbb{R}^3 \times (0,\infty)) \\ \bar{u} = \bar{g}, \bar{u}_t = \bar{h} & (\text{on } \mathbb{R}^3 \times \{t=0\}), \end{cases}$$

where we define $\bar{g}$ and $\bar{h}$ analogously. For $x = (x_1,x_2) \in \mathbb{R}^2$, write $\bar{x} = (x_1,x_2,0) \in \mathbb{R}^3$. Then by Kirchoff's formula, one has

$$u(x,t) = \bar{u}(\bar{x},t) = \frac{\partial}{\partial t}\left(t\fint_{\partial\bar{B}(\bar{x},t)} \bar{g}\,d\bar{S} \right) + t\fint_{\partial\bar{B}(\bar{x},t)} \bar{h}\,d\bar{S},$$

where $\bar{B}(\bar{x},t)$ denotes the ball in $\mathbb{R}^3$ with center $\bar{x}$, radius $t>0$, and where $d\bar{S}$ denotes the 2-dimensional surface measure on $\partial\bar{B}(\bar{x},t)$. But we can write

$$\fint_{\partial\bar{B}(\bar{x},t)} \bar{g}\,d\bar{S} = \frac{1}{4\pi t^2}\int_{\partial\bar{B}(\bar{x},t)} \bar{g}\,d\bar{S} = \frac{2}{4\pi t^2}\int_{B(x,t)} g(y)(1+|D\gamma(y)|^2)^{1/2}\,dy,$$

where $\gamma(y) = (t^2 - |y-x|^2)^{1/2}$ for $y \in B(x,t)$. The factor 2 comes from the two hemispheres of $\partial\bar{B}(\bar{x},t)$. Since $(1+|D\gamma|^2)^{1/2} = t(t^2-|y-x|^2)^{-1/2}$, one has

$$\begin{aligned} \fint_{\partial\bar{B}(\bar{x},t)} \bar{g}\,d\bar{S} &= \frac{1}{2\pi t}\int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy \\ &= \frac{t}{2}\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy. \end{aligned}$$

Using this and the coarea formula, one obtains

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\,dy\right) + \frac{t^2}{2}\fint_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy.$$

Using the same reasoning as above to pass the derivative inside the integral, one has

$$\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\,dy\right) = t\fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\,dy + t\fint_{B(x,t)} \frac{Dg(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\,dy.$$

Thus for $n = 2$, we obtain **Poisson's Formula**,

$$\boxed{u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{tg(y) + t^2 h(y) + tDg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,dy \qquad (x\in\mathbb{R}^2, t>0)} \tag{1.14}$$

Notice that in solving the wave equation for $n = 2$, we used the *method of descent*; i.e., we solved the problem for $n = 3$ first, and then dropped down to $n = 2$.

General Solutions for $n \geq 2$ Following similar extensive calculations as above (see Appendix for the work), we obtain the following solutions to the wave equation,

$$\boxed{u(x,t) = \begin{cases} \dfrac{1}{\gamma(n)}\left[\left(\dfrac{\partial}{\partial t}\right)\left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} g\,dS\right) + \left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} h\,dS\right)\right] & \begin{matrix} n\geq 3 \\ n \text{ odd}\end{matrix} \\ \dfrac{1}{\beta(n)}\left[\left(\dfrac{\partial}{\partial t}\right)\left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \dfrac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\right. & \\ \qquad\qquad \left. + \left(\dfrac{1}{t}\dfrac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \dfrac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\right] & \begin{matrix} n\geq 2 \\ n \text{ even}\end{matrix} \end{cases}} \tag{1.15}$$

where in the first equation $\gamma_n = 1\cdot 3\cdot\dots\cdot(n-2)$, and $\beta(n) = 2\cdot 4\cdot\dots\cdot(n-2)\cdot n$. The above solutions highlight an interesting contrast between odd-dimensional, and even-dimensional solutions. Note that if n is odd and $n \geq 3$, the initial data at a point $x \in \mathbb{R}^n$ only affect the solution u along the boundary $\{(y,t) \mid t > 0, |x-y| = t\}$ of the cone $C = \{(y,t) \mid t > 0, |x-y| < t\}$. On the other hand, if n is given, the initial data can affect the solution within all of C. Thus, a disturbance originating at x propagates along a sharp wavefront in odd dimensions, but continues to have effects even after the leading edge of the wavefront passes in even dimensions. This phenomenon is **Huygen's principle**.

1.2 Solution to the Nonomgeneous Wave Equation

Now we will use the solutions from the previous sections to address the nonhomogeneous wave equation,

$$\begin{cases} u_{tt} - \Delta u = f & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = 0, u_t = 0 & (\text{on } \mathbb{R}^n \times \{t=0\}). \end{cases} \tag{1.16}$$

To solve this problem, we use Duhamel's principle[1] by setting $u = u(x,t;s)$ to be the solution of

$$\begin{cases} u_{tt}(\cdot;s) - \Delta u(\cdot,s) \quad = 0 & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u(\cdot;s) = 0, u_t(\cdot;s) = f(\cdot,s) & (\text{on } \mathbb{R}^n \times \{t=0\}). \end{cases} \tag{1.17}$$

Thus, by Duhamel's principle,

$$u(x,t) \coloneqq \int_0^t u(x,t;s)\,ds \qquad (x\in\mathbb{R}^n, t\geq 0) \tag{1.18}$$

[1] See https://en.wikipedia.org/wiki/Duhamel%27s_principle for an explanation.

is a solution of the above nonhomogeneous wave equation. Consequently, the solution to the general inhomogeneous wave problem,

$$\begin{cases} u_{tt} - \Delta u = f & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R}^n \times \{t=0\}) \end{cases} \tag{1.19}$$

is equal to the sum of the solutions to the problems

$$\begin{cases} u_{tt} - \Delta u = 0 & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R}^n \times \{t=0\}) \end{cases} \quad \text{and} \quad \begin{cases} u_{tt} - \Delta u = f & (\text{in } \mathbb{R}^n \times (0,\infty)) \\ u = 0, u_t = 0 & (\text{on } \mathbb{R}^n \times \{t=0\}) \end{cases} \tag{1.20}$$

1.2.1 Example 1: Solution to Nonhomogeneous Problem for $n = 1$

By d'Alembert's formula, one has

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t+s} f(y,s)\,dy.$$

Thus,

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t+s} f(y,s)\,dyds = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds \qquad (x \in \mathbb{R}, t \geq 0).$$

1.2.2 Example 2: Solution to Nonhomogeneous Problem for $n = 3$

By Kirchoff's formula, one has

$$u(x,t;s) = (t-s)⨍_{\partial B(x,t-s)} f(y,s)\,dS.$$

This implies that

$$\begin{aligned} u(x,t) &= \int_0^t (t-s)⨍_{\partial B(x,t-s)} f(y,s)\,dSds \\ &= \int_0^t (t-s)\frac{1}{4\pi(t-s)^2}\int_{\partial B(x,t-s)} f(y,s)\,dSds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\,dSds \\ &= \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dSdr. \end{aligned}$$

Hence, we conclude that

$$u(x,t) = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy \qquad (x \in \mathbb{R}^3, t \geq 0),$$

where the integrand on the right is called a *retarded potential*.

1.3 An Introduction to Energy Estimates

In this section, we will introduce the concept of *energy* for the initial value problems for the wave equation. Subsequently, we will use the energy to establish some fundamental properties of solutions to the wave equation. In particular, we will show uniqueness of solution, and finite propagation speed. Our standing notation for this section is as follows. We will be working with a bounded, open subset $U \subset \mathbb{R}^n$ with a smooth boundary ∂U, and the sets $U_T = U \times (0,T]$, $\Gamma_t = \bar{U}_t \setminus U_T$, where $T > 0$. We will also be looking at the general initial value problem,

$$\begin{cases} u_t t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } U \times \{t=0\}. \end{cases} \tag{1.21}$$

Theorem 1.2 (Uniqueness for the Wave Equation). *There exists at most one function $u \in C^2(\bar{U}_t)$ solving (1.21).*

Proof. Suppose $\widetilde{u}$ is another solution in $C^2(\bar{U}_t)$ to the given initial value problem. Then $w \coloneqq u - \widetilde{u}$ solves the following initial value problem

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T \\ w_t = 0 & \text{on } U \times \{t = 0\}. \end{cases}$$

Define the energy,

$$e(t) \coloneqq \frac{1}{2} \int_U w_t^2(x,t) + |Dw(x,t)|^2 \, dx \qquad (0 \leq t \leq T).$$

Since U was assumed to be a bounded set, and $w \in C^2(\bar{U}_T)$, for all $0 \leq t \leq T$, there exist constants M and N so that for all $x \in U$, $|w_t(x,t)|^2 \leq M\chi_U$ and $|Dw(x,t)|^2 \leq N\chi_U$, both which are integrable functions since U is bounded. Thus by the Dominated Convergence Theorem, when differentiating $e(t)$ with respect to t, we can pass the time derivative inside the integral sign to obtain the following:

$$\begin{aligned} \dot{e}(t) &= \int_U w_t w_{tt} + Dw(x,t) \cdot Dw_t(x,t) \, dx \\ &= \int_U w_t(w_{tt} - \Delta w) \, dx = 0. \end{aligned}$$

Note that there is no boundary term since $w = 0$, which means $w_t = 0$, on $\partial U \times [0,T]$. Therefore, for all $0 \leq t \leq T$, $e(t) = e(0) = 0$, so that $w_t, Dw \equiv 0$ within U_T. Since $w \equiv 0$ on $U \times \{t = 0\}$, we conclude that $w = u - \widetilde{u} \equiv 0$ in U_T. The proof concludes. □

Now we will use the energy method to prove another distinct property of hPDE, the *finite propagation speed*. First, we define what we mean by the **domain of dependence**.

Definition 1.2 (Domain of Dependence). *Suppose $u \in C^2(U)$ solves the wave equation*

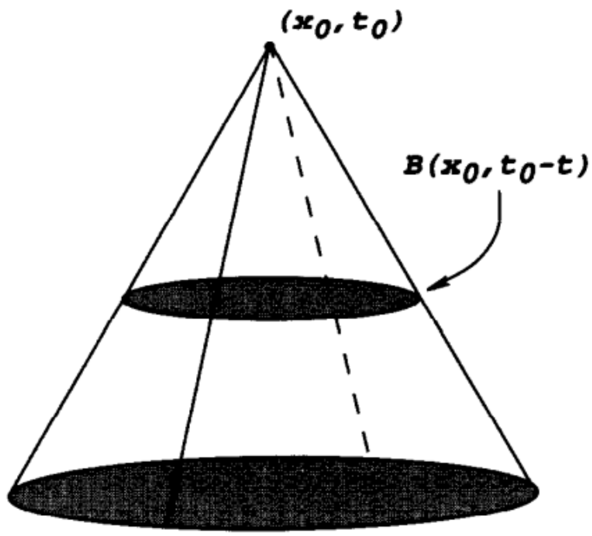
$$u_{tt} - \Delta u = 0 \qquad (\text{in } \mathbb{R}^n \times (0,\infty)).$$

Fix $x_0 \in \mathbb{R}^n$ and $t_0 > 0$. Then define the cone

$$C \coloneqq \{(x,t) \mid 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}. \tag{1.22}$$

In some sense, we have already seen this explicitly in the representation formulas for solutions to the wave equation. However, we will now use energy methods. We have the following theorem.

Theorem 1.3 (Finite Propagation Speed). *If $u \equiv u_t \equiv 0$ on $B(x_0, t_0)$, then $u \equiv 0$ within the cone C. I.e., any disturbance originating outside $B(x_0, t_0)$ has no effect on the solution within C, and consequently has finite propagation speed.*



Cone of dependence

Figure 1: Depiction of the domain of dependence for a point (x_0, t_0). I.e., the value of u depends only on the points contained inside this cone.

1.4 Exercises

Problem 1-1.

(a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using the change of variables $\xi = x + t$, $\eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a) and (b) to rederive d'Alembert's formula.

(a) Consider the PDE $u_{xy} = 0$. Let $v(x,y) = u_x$. Then v satisfies the differential equation $v_y = 0$. Thus since v is independent of y, one has $u_x(x,y) = v(x,y) = f(x)$ for some arbitrary function f. But then integrating both sides with respect to x, and supposing that $F(x)$ is the antiderivative of $f(x)$,

$$u(x,y) = F(x) + G(y),$$

where G is some arbitrary function of y.

(b) Define $\xi = x + t$ and $\eta = x - t$ so that $x = (\xi + \eta)/2$ and $t = (\xi - \eta)/2$. Then,

$$\begin{aligned} u_\xi &= u_x \frac{\partial x}{\partial \xi} + u_t \frac{\partial t}{\partial \xi} \\ &= \frac{1}{2} u_x + \frac{1}{2} u_t. \\ u_{\xi\eta} &= \frac{1}{2}\left[u_{xx}\frac{\partial x}{\partial \eta} + u_{xt}\frac{\partial t}{\partial \eta}\right] + \frac{1}{2}\left[u_{tx}\frac{\partial x}{\partial \eta} + u_{tt}\frac{\partial t}{\partial \eta}\right] \\ &= \frac{1}{2}[u_{xx} - u_{xt}] + \frac{1}{2}[u_{tx} - u_{tt}] = \frac{1}{2}[u_{tt} - u_{xx}], \end{aligned}$$

where we used the fact that $u_{xt} = u_{tx}$. Thus, the proof concludes.

(c) Consider the initial value problem,

$$\begin{cases} u_{tt} - u_{xx} = 0 & (\text{in } \mathbb{R} \times (0, \infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R} \times \{t = 0\}). \end{cases}$$

Define the change of variables $\xi = x + t$ and $\eta = x - t$ as above. By (b), $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$. By (a), it follows that u must be of the form,

$$u(\xi, \eta) = F(\xi) + G(\eta).$$

Thus, $u(x,t) = F(x+t) + G(x-t)$. Now we must match the initial conditions. At $t = 0$,

$$u(x,0) = F(x) + G(x) = g(x).$$
$$u_t(x,0) = F'(x) - G'(x) = h(x).$$

Differentiating the first equation, one first obtains the system

$$\begin{cases} F'(x) + G'(x) = g'(x) \\ F'(x) - G'(x) = h(x). \end{cases}$$

Thus, we get

$$F'(x) = \frac{1}{2}(g'(x) + h(x)) \implies F(x+t) = \frac{1}{2}[g(x+t) - g(0)] + \frac{1}{2}\int_0^{x+t} h(y)\, dy + F(0).$$

Likewise, we get

$$G(x-t) = \frac{1}{2}[g(x-t) - g(0)] - \int_0^{x-t} h(y)\, dy + G(0).$$

Adding the two equations, and using the fact that $F(0) + G(0) = g(0)$, we conclude that

$$u(x,t) = \frac{1}{2}[g(x-t) + g(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy,$$

which is precisely d'Alembert's formula.

Problem 1-2. Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations. Show

$$u_{tt} - \Delta u = 0$$

where $u = E^i$ or B^i $(i = 1, 2, 3)$.

Recall that Maxwell's equations are given by the following system,

$$\begin{cases} \mathbf{E}_t = \nabla \times \mathbf{B} \\ \mathbf{B}_t = -\nabla \times \mathbf{E} \\ \nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{E} = 0. \end{cases}$$

Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations. Then differentiating the first equation and using the fact that time derivatives commute with spatial derivatives, one obtains

$$\mathbf{E}_{tt} = \nabla \times \mathbf{B}_t = -\nabla \times (\nabla \times E) = -\nabla(\nabla \cdot \mathbf{E}) + \Delta\mathbf{E} = \Delta\mathbf{E},$$

where the final equality follows from the fact that $\nabla \cdot \mathbf{E} = 0$. Thus, we have

$$\mathbf{E}_{tt} - \Delta\mathbf{E} = 0 \iff (E^1_{tt} - \Delta E^1, \ldots, E^3_{tt} - \Delta E^3) = \mathbf{0}.$$

Hence, each E^i satisfies the wave equation. Now differentiate the second equation. We get

$$\mathbf{B}_{tt} = -\nabla \times \mathbf{E}_t = -\nabla \times (\nabla \times \mathbf{B}) = -\nabla(\nabla \cdot \mathbf{B}) + \Delta\mathbf{B} = \Delta\mathbf{B}.$$

Thus we have,

$$\mathbf{B}_{tt} - \Delta\mathbf{B} = 0 \iff (B^1_{tt} - \Delta B^1, \ldots, B^3_{tt} - \Delta B^3) = \mathbf{0}.$$

Therefore, each B^i satisfies the wave equation.

Problem 1-3. (Equipartition of Energy). Let $u \in C^2(\mathbb{R} \times [0,\infty))$ solve the initial-value problem for the wave equation in one-dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & (\text{in } \mathbb{R} \times (0,\infty)) \\ u = g, u_t = h & (\text{on } \mathbb{R} \times \{t = 0\}). \end{cases}$$

Suppose g, h have compact support. The *kinetic energy* is $k(t) \coloneqq \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and the *potential energy* is $p(t) \coloneqq \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$. Prove

(i) $k(t) + p(t)$ is constant in t,

(ii) $k(t) = p(t)$ for all large enough times t.

(i) Let us define the kinetic and potential energy, $k(t)$ and $p(t)$ as given above. We wish to show that the total energy
$$e(t) = k(t) + p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 + u_x^2\,dx$$
is conserved, where u is given by d'Alembert's formula,
$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$
so that
$$u_t(x,t) = \frac{1}{2}[g'(x+t) - g'(x-t) + h(x+t) + h(x-t)].$$
$$u_x(x,t) = \frac{1}{2}[g'(x+t) + g'(x-t) + h(x+t) - h(x-t)].$$
Let $T > 0$, and let $K \subset \mathbb{R}$ be a compact set large enough so that $\operatorname{supp} g, \operatorname{supp} h$ are both contained in K. Then it follows that u_t^2 and u_x^2 are both supported in $K_T \coloneqq K + [-T,T]$ for all $0 \le t \le T$. Since u_t^2 and u_x^2 are continuous on $\mathbb{R}$ (and thus on K_T) for all $0 \le t \le T$, by the extreme value theorem, there exist finite constants M and N so that for all $x \in K$ and $0 \le t \le T$, $|u_t|^2 \le M\chi_{K_T}$ and $|u_x^2| \le N\chi_{K_T}$, where both dominating functions are integrable since K_T is compact. Thus using the Dominated Convergence Theorem, taking the derivative of $e(t)$ gives us,
$$\begin{aligned} \dot{e}(t) &= \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u_t^2 + u_x^2\,dx = \frac{1}{2}\frac{\partial}{\partial t}\int_{K_T} u_t^2 + u_x^2\,dx \\ &= \frac{1}{2}\int_{K_T} \partial_t(u_t^2 + u_x^2)\,dx \\ &= \int_{K_T} u_t u_{tt} + u_x \cdot \partial_x u_t\,dx \\ &= \int_{K_T} u_t(u_{tt} - u_{xx})\,dx \qquad \text{(by integration by parts)} \\ &= 0, \end{aligned}$$
where there is no boundary term since u_x, u_t are supported inside K_T. So we have shown that $\dot{e}(t) = 0$ for all $0 \le t \le T$. Since $T > 0$ was chosen arbitrarily, it follows that $\dot{e}(t) \equiv 0$. Therefore, the total energy is constant in t.

(ii) What we wish is that in the limit $t \to \infty$,
$$\left|\int_{\mathbb{R}} u_t^2 - u_x^2\,dx\right| \to 0.$$
First, we compute
$$u_t^2 - u_x^2 = h(x+t)h(x-t) - h(x+t)g'(x-t) + h(x-t)g'(x+t) - g'(x-t)g'(x+t).$$

Then

$$\left|\int_{\mathbb{R}} u_t^2 - u_x^2 \, dx\right| \le \int_{-\infty}^{\infty} |h(y)h(y-2t)| \, dy + \int_{-\infty}^{\infty} |h(y)g'(y-2t)| \, dy$$
$$+ \int_{-\infty}^{\infty} |h(y-2t)g'(y)| \, dy + \int_{-\infty}^{\infty} |g'(y-2t)g'(y)| \, dy.$$

Let $K \subset \mathbb{R}$ be a compact set large enough that contains the support of both g and h. Then we may write,

$$\left|\int_{\mathbb{R}} u_t^2 - u_x^2 \, dx\right| \le \int_K |h(y)h(y-2t)| \, dy + \int_K |h(y)g'(y-2t)| \, dy$$
$$+ \int_K |h(y-2t)g'(y)| \, dy + \int_K |g'(y-2t)g'(y)| \, dy.$$

We will only show that the first term on the right vanishes since the work for the remaining terms is identical. Since $h(u) \to 0$ as $|u| \to \infty$ (by compact support of h), we know that pointwise $|h(y)h(y-2t)| \to 0$ as $t \to \infty$. Next since h is continuous, there exists some finite $M > 0$ so that $|h(y)|, |h(y-2t)| \le M$ for all $y \in K$. Thus, $|h(y)h(y-2t)| \le M^2 \chi_K \in L^1(\mathbb{R})$ since K is compact. Hence, by the Dominated Convergence Theorem, one has that the first term vanishes as $t \to \infty$. As we stated before, the remaining three terms also vanish as $t \to \infty$. Thus, we conclude that as $t \to \infty$,

$$\left|\int_{\mathbb{R}} u_t^2 - u_x^2 \, dx\right| \to 0.$$

Problem 12-6. Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^3 \times \{t=0\}, \end{cases}$$

where g, h are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \le C/t \qquad (x \in \mathbb{R}^3, t > 0).$$

Let u be the solution to the initial value problem given above. We recall that u is given by Kirchoff's formula,

$$u(x,t) = ⨍_{\partial B(x,t)} th(y) + g(y) + Dg(y) \cdot (y-x) \, dS(y) \qquad (x \in \mathbb{R}^3, t > 0)$$

Let $R > 0$ be a fixed positive constant so that the support of g and h are both contained inside the ball $B(0,R)$. We can rewrite,

$$|u(x,t)| \le \frac{1}{4\pi t^2}\left[\underbrace{\int_{\partial B(x,t)} t|h(y)| \, dS(y)}_{(a)} + \underbrace{\int_{\partial B(x,t)} |g(y)| \, dS(y)}_{(b)} + \underbrace{\int_{\partial B(x,t)} |Dg(y)||y-x| \, dS(y)}_{(c)}\right].$$

We will try bounding each term independently.

(a) Let us denote $\Sigma_t = \partial B(x,t) \cap B(0,R)$. Then it follows that

$$\int_{\partial B(x,t)} t|h(y)| \, dS(y) = t\int_{\Sigma_t} |h(y)| \, dS(y) \le tM \cdot 4\pi R^2,$$

where $M > 0$ is a constant so that $|h| \le M$, and we used the fact that the surface area of $\partial B(x,t) \cap B(0,R)$ cannot exceed the surface area of $4\pi R^2$.[2]

[2] https://math.stackexchange.com/questions/1302682/2d-wave-equation-decay-estimate

(b) Now let $N > 0$ so that $|g| \leq N$. Then repeating the work from above shows that

$$\int_{\partial B(x,t)} |g(y)|\, dS(y) \leq N \cdot 4\pi R^2.$$

(c) Now we note that for all $y \in \partial B(x,t)$, $|y-x| = t$. So taking $L > 0$ so that $|Dg| \leq L$, and repeating the work from (a), one has that

$$\int_{\partial B(x,t)} |Dg(y)||y-x|\, dS(y) = tL \cdot 4\pi R^2.$$

Combining all of the pieces together, we find that

$$\begin{aligned} |u(x,t)| &\leq \frac{1}{4\pi t^2} \cdot \left[4\pi R^2 (M+L)t + 4\pi R^2 N\right] \\ &\leq \frac{1}{t^2} \cdot C'(1+t) \leq \frac{C}{t} \end{aligned}$$

Problem 12-5. Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where g, h have compact support. Show that

$$|u(x,t)| \leq C/t^{\frac{1}{2}} \qquad (x \in \mathbb{R}^2, t > 0)$$

for some constant C.

Let u be a solution to the initial value problem for the wave equation on $\mathbb{R}^2$, where the initial data are compactly supported. We recall that u is given by Poisson's Formula,

$$u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{tg(y) + t^2 h(y) + tDg(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\, dy \qquad (x \in \mathbb{R}^2, t > 0).$$

Let $R > 0$ be large enough so that the ball $B(0,R)$ contains the support of both g and h. We begin by writing,

$$u(x,t) = \frac{1}{2\pi t^2}\left[\underbrace{\int_{B(x,t)} \frac{tg(y)}{(t^2-|y-x|^2)^{1/2}}\, dy}_{(a)} + \underbrace{\int_{B(x,t)} \frac{t^2 h(y)}{(t^2-|y-x|^2)^{1/2}}\, dy}_{(b)} + \underbrace{\int_{B(x,t)} \frac{tDg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\, dy}_{(c)}\right].$$

We will look at each of the above terms independently.

(a) Since the support of g is contained in $B(0,R)$, we can write

$$\begin{aligned} \int_{B(x,t)} \frac{t|g(y)|}{(t^2-|y-x|^2)^{1/2}}\, dy &= \int_{B(x,t)\cap B(0,R)} \frac{t|g(y)|}{(t^2-|y-x|^2)^{1/2}}\, dy \\ &\leq \int_{B(x,t)\cap B(0,R)} \frac{tM}{(t+|y-x|)^{1/2}(t-|y-x|)^{1/2}}\, dy \\ &\leq \int_{B(x,t)\cap B(0,R)} \frac{t^{1/2}M}{(t-|y-x|)^{1/2}}\, dy \\ &\leq \int_{B(0,R)} \frac{t^{1/2}M}{(t-|y-x|)^{1/2}}\, dy =: (\star). \end{aligned}$$

where $M > 0$ is chosen such that $|g(y)| \leq M$ for all $y \in B(0,R)$. By the triangle inequality, one has $|y-x| \leq R + |x|$ so that [!! Complete Later !!]

Problem 1-5. (Stokes' Rule) Assume u solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0, u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Show that $v \coloneqq u_t$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = h, v_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

This is *Stokes' rule*.

Assume u solves the given initial value problem, and set $v \coloneqq u_t$. The boundary value conditions for v are immediately clear by the initial value problem for u. Thus, it suffices to show that $\square v = 0$. Observe the following:

$$\begin{aligned} v_{tt} &= u_{ttt}. \\ \Delta v &= \Delta(u_t) = \partial_t(\Delta u) = \partial_t(u_{tt}) = u_{ttt}. \end{aligned}$$

This implies that $v_{tt} - \Delta v = u_{ttt} - u_{ttt} = 0$. Thus the proof concludes.

2 General Analytic Techniques

In this section, we will revisit the linear wave equation. This time, however, we will be doing so from the lens of Fourier/harmonic analysis, and functional analysis.

2.1 Hölder Spaces

Definition 2.1. *Hölder Continuous Function with Exponent γ A function u on U is said to be **Hölder continuous with exponent** $0 < \gamma \leq 1$ and a constant C if for any $x, y \in U$, $|u(x) - u(y)| \leq C|x-y|^\gamma$.*

Definition 2.2 (Hölder Norms).
(i) If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\bar{U})} \coloneqq \sup_{x \in U} |u(x)|.$$

(ii) We define the γth-Hölder seminorm of $u : U \to \mathbb{R}$ to be the following

$$[u]_{C^{0,\gamma}(\bar{U})} \coloneqq \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\gamma} \right\},$$

and the γth-Hölder norm to be

$$\|u\|_{C^{0,\gamma}(\bar{U})} \coloneqq \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}.$$

This allows to define the Hölder space, $C^{k,\gamma}(\bar{U})$ as the following.

Definition 2.3 (Hölder Space). *The **Hölder space** $C^{k,\gamma}(\bar{U})$ consists of all functions $u \in C^k(\bar{U})$ for which the norm*

$$\|u\|_{C^{k,\gamma}(\bar{U})} \coloneqq \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^k \alpha]_{C^{0,\gamma}(\bar{U})} < \infty.$$

In particular, $\|\cdot\|_{C^{k,\gamma}(\bar{U})}$ forms a norm on $C^{k,\gamma}(\bar{U})$, and $C^{k,\gamma}(\bar{U})$ is a Banach space under this norm.

2.2 Sobolev Spaces

While the classes of Hölder functions on U are nice function spaces, often times, we cannot make strong enough analytic estimates on solutions obtained for PDE to prove that they belong to such spaces; by our definitions above, we require a certain degree of regularity. Therefore, we now seek to replace these spaces with function spaces that have some, but not a lot, of smoothness properties. We start by defining the notion of a *weak derivative*.

Definition 2.4 (Space of Test Functions). *Let U be an open subset of $\mathbb{R}^n$. Then we define the space $C_c^\infty(U)$ to be the space of infinitely differentiable (i.e., smooth) functions $\phi : U \to \mathbb{R}$ with compact support in U. We call such functions **test functions**.*

Definition 2.5 (Weak Derivative). *Suppose $u, v \in L^1_{loc}(U)$ (recall that $L^1_{loc}(U)$ denotes the space of locally integrable functions on U). Then for any multiindex α, we say that v is the α^{th} **weak partial derivative of** u, denoted $D^\alpha u = v$, provided that*

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v\phi \, dx, \qquad \forall \phi \in C_c^\infty(U). \tag{2.1}$$

The motivation behind weak derivatives comes from the integration by parts formula. Suppose $u \in C^1(U)$. Then by integration by parts, for any $\phi \in C_c^\infty(U)$, one has

$$\int_U u\phi_{x_i} \, dx = -\int_U u_{x_i} \phi \, dx \qquad (i = 1, 2, \ldots, n),$$

where there are no boundary terms since ϕ hs compact support in U. More generally, if $u \in C^k(U)$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, then

$$\int_U u D^\alpha \phi \, dx = -\int_U D^\alpha u \phi \, dx.$$

If u is not k-times continuously differentiable, then the left side is still well-defined so long as u is locally integrable in U. But the right side is generally not defined. Thus, we instead look for locally integrable functions v so that the above formula holds, with v taking on the role of $D^\alpha u$.

Example 2.1 (Example of a Function with a Weak Derivative). *Let $n = 1$, $U = (0, 2)$, and*

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 \leq x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

We claim that v is the weak partial derivative of u. Indeed, let ϕ be any compactly supported smooth function in U. Then

$$\begin{aligned} \int_0^2 u\phi' \, dx &= \int_0^1 x\phi' \, dx + \int_1^2 \phi' \, dx \\ &= -\int_0^1 \phi(x) \, dx + \phi(1) - \phi(1) = -\int_0^2 v\phi \, dx. \end{aligned}$$

Example 2.2 (Example of a Function without Weak Derivative). *Let $n = 1$, $U = (0, 2)$, and*

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 \leq x < 2. \end{cases}$$

Assume to the contrary that there exists some locally integrable function $v \in L^1_{loc}(U)$ such that

$$\int_0^2 u\phi' \, dx = -\int_0^2 v\phi \, dx$$

for all test functions ϕ. Then some quick computations give us,

$$\begin{aligned} -\int_0^2 v\phi \, dx &= \int_0^2 u\phi' \, dx \\ &= \int_0^1 x\phi' \, dx + \int_1^2 2\phi'(x) \, dx \\ &= \phi(1) - \int_0^1 \phi \, dx - 2\phi(1) = -\int_0^1 \phi \, dx - \phi(1). \end{aligned}$$

Now choose a sequence $\{\phi_m\}_{m=1}^\infty$ of smooth functions satisfying

$$0 \leq \phi_m \leq 1, \quad \phi_m(1) = 1, \quad \phi_m(x) \to 0 \text{ for all } x \neq 1.$$

By the above computations, one has

$$1 = \phi_m(1) = \int_0^2 v\phi_m \, dx - \int_0^1 \phi \, dx.$$

Thus taking the limit $m \to \infty$, one has

$$1 = \lim_{m\to\infty} \phi_m(1) = \lim_{m\to\infty} \left[\int_0^2 v\phi_m \, dx - \int_0^1 \phi_m \, dx \right] = 0,$$

which is a contradiction. Thus u has no weak partial derivatives.

More fundamentally, if a locally integrable function has a weak partial derivative, then it must be unique as the following lemma asserts.

Lemma 2.1 (Uniqueness of Weak Derivatives)**.** *Assume $v, \widetilde{v} \in L^1_{loc}(U)$ are weak partial derivatives of $u \in L^1_{loc}(U)$. Then for any $\phi \in C_c^\infty(U)$,*

$$\begin{aligned}\int_U (v - \widetilde{v})\phi \, dx &= \int_U v\phi \, dx - \int_U \widetilde{v}\phi \, dx \\ &= (-1)^{|\alpha|} \int_U uD^\alpha \phi \, dx - (-1)^{|\alpha|} \int_U uD^\alpha \phi \, dx = 0.\end{aligned}$$

Thus, $v = \widetilde{v}$ a.e.

2.2.1 Definition and Basic Properties of Sobolev Spaces

Keeping the notion of weak partial derivatives in mind, we now make the following definition of **Sobolev spaces**

Definition 2.6 (The Sobolev Space $W^{k,p}(U)$)**.** *The* ***Sobolev space*** *$W^{k,p}(U)$ consists of all locally integrable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.*

As a function space, two functions in $W^{k,p}(U)$ are said to be the "same" if they agree a.e (which is precisely how we view functions in the L^p spaces). Moreover, in the case that $p = 2$, since L^2 is a Hilbert space, we denote $H^k(U) \coloneqq W^{k,2}(U)$. Now we introduce a norm on our Sobolev spaces.

Definition 2.7 (Sobolev Norm)**.** *If $u \in W^{k,p}(U)$, we define its norm to be*

$$\|u\|_{W^{k,p}} \coloneqq \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p \, dx \right)^{1/p} & (1 \le p < \infty) \\ \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(U)} & (p = \infty) \end{cases} \tag{2.2}$$

Notice the resemblance to the L^p-norms. In particular, we define convergence in the Sobolev spaces as follows:

Definition 2.8 (Convergence in the Sobolev Spaces)**.**

(i) Let $\{u_m\}_{m=1}^\infty, u \in W^{k,p}(U)$. We say that u_m converges to u in $W^{k,p}$ provided

$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(U)} = 0. \tag{2.3}$$

(ii) We say that $u_m \to u$ in $W^{k,p}_{loc}(U)$ iff $\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(V)} = 0$ for all $V \Subset U$.

(iii) (Closure of Test Functions) We denote by $W_0^{k,p}(U)$ the closure of the space of test functions, $C_c^\infty(U)$ in $W^{k,p}(U)$.

Example 2.3 (Verifying a Function is in a Sobolev Space)**.** *Let $U = B^0(0,1)$ denote the open unit ball in $\mathbb{R}^n$, and let $u(x) = |x|^{-\alpha}$, where $x \in U \setminus \{0\}$. We will calculate the values of α, n, and p for which $u \in W^{1,p}(U)$. The "problematic region" is at $x = 0$, where u is not differentiable in the classical sense. Therefore, we will isolate this problem to a small ε-ball. Indeed, since u is differentiable on $U \setminus B(0,\varepsilon)$, one has that for any fixed $\phi \in C_c^\infty(U)$ and fixed $\varepsilon > 0$,*

$$\int_{U \setminus B(0,\varepsilon)} u\phi_{x_i} \, dx = -\int_{U \setminus B(0,\varepsilon)} u_{x_i}\phi \, dx + \int_{\partial B(0,\varepsilon)} u\phi\nu^i \, dS,$$

where $\nu = (\nu^1, \dots, \nu^n)$ denotes the inward pointing normal on $\partial B(0,\varepsilon)$. Note that

$$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \qquad (x \neq 0).$$

If $\alpha+1<n$, then $|Du(x)| \in L^1(U)$. We will now use this to force $\varepsilon \to 0$. Indeed, one first has

$$\left|\int_{\partial B(0,\varepsilon)} u\phi\nu^i \, dS\right| \le \int_{\partial B(0,\varepsilon)} |u\|\phi| \, dS = \int_{\partial B(0,\varepsilon)} \varepsilon^{-\alpha}|\phi| \, dS \le \|\phi\|_{L^\infty(\partial B(0,\varepsilon))} \int_{\partial B(0,\varepsilon)} \varepsilon^{-\alpha} \, dS$$
$$\le C\varepsilon^{n-1-\alpha} \xrightarrow{\varepsilon\to 0} 0.$$

In going to the final inequality, we used the fact that since $\partial B(0,\varepsilon) \subset U$ for sufficiently small ε, $\|\phi\|_{L^\infty(\partial B(0,\varepsilon))} \le \|\phi\|_{L^\infty(U)}$, which is a constant. Thus we get

$$\int_U u\phi_{x_i} \, dx = -\int_U u_{x_i}\phi \, dx$$

for all $\phi \in C_c^\infty(U)$ provided $0 \le \alpha \le n-1$. Furthermore, since $|Du(x)| \in L^p(U)$ if and only if $(\alpha+1)p < n$, we conclude that $u \in W^{1,p}(U)$ if and only if $a < \frac{n-p}{p}$. In particular, $u \notin W^{1,p}(U)$ for each $p \ge n$.

2.3 Fourier Theory

We start by considering the *Schwartz class of functions* on $\mathbb{R}^n$.

Definition 2.9 (Schwartz Class of Functions)**.** *By $\mathcal{S}(\mathbb{R}^n)$, we denote the Schwarz class of functions,*

$$\mathcal{S}(\mathbb{R}^n) \coloneqq \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}_0^n, \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \right\}. \tag{2.4}$$

I.e., the Schwartz class of functions consists of all the smooth functions on $\mathbb{R}^n$ that decay faster than any polynomial near ∞. We introduce the seminorms,

$$\|f\|_{\alpha,\beta} \coloneqq \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$$

on this space, and we say that a sequence $\{u_n\} \subset \mathcal{S}(\mathbb{R}^n)$ converges to $u \in \mathcal{S}(\mathbb{R}^n)$ if and only if for all α, β, $\|u_n - u\|_{\alpha,\beta} \xrightarrow{n\to\infty} 0$. In particular, $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $L^p(\mathbb{R}^n)$ for all $p \ge 1$. Thus we define the Fourier transform as follows.

Definition 2.10 ((Inverse) Fourier Transform)**.** *Given a function $f \in \mathcal{S}(\mathbb{R}^n)$, define its Fourier transform by the formula*

$$\hat{f}(\xi) \coloneqq \int_{\mathbb{R}^n} f(x)e^{-2\pi i(x,\xi)} \, dx. \tag{2.5}$$

Conversely, we define the inverse Fourier transform by the formula,

$$\check{f}(x) \coloneqq \int_{\mathbb{R}^n} f(x)e^{2\pi i(x,\xi)} \, d\xi. \tag{2.6}$$

The Fourier transform has the following properties:

Theorem 2.1 (Properties of the Fourier Transform)**.**

(1) The Fourier transform maps the Schwartz space into the Schwartz space.

(2) (Plancherel's Theorem) For $f \in \mathcal{S}(\mathbb{R}^n)$, $\|f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$. As a consequence, the Fourier transform extends to an isometric automorphism on $L^2(\mathbb{R}^n)$. (See Appendix A for a proof of this statement.)

As we will see now, the Fourier transform is a powerful tool in PDE.

2.3.1 Solving the Wave Equation using Fourier Transforms

Consider the initial value problem again:

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \phi = f, \phi_t = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

where $f, g \in \mathcal{S}(\mathbb{R}^n)$. Using the basic properties of the Fourier transform, applying the Fourier transform to the wave equation yields,

$$-\partial_t^2 \hat{\phi}(t,\xi) - 4\pi^2|\xi|^2\hat{\phi}(t,\xi) = 0,$$

which is an ODE. Solving this ODE, and using the initial value conditions, one obtains the solution

$$\hat{\phi}(\xi,t) = \frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi t|\xi|) + \hat{f}(\xi)\cos(2\pi t|\xi|).$$

In particular, the solution is unique and Schwartz in ξ. Therefore, taking the inverse Fourier transform, and using the fact that the inverse Fourier transform is unique up to null sets, we obtain the unique smooth representation formula,

$$\boxed{\phi(t,x) = \int_{\mathbb{R}^n} \left[\frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi t|\xi|) + \hat{f}(\xi)\cos(2\pi t|\xi|)\right] e^{2\pi i(x,\xi)}\, d\xi}. \tag{2.7}$$

Thus we have the following result.

Theorem 2.2 (Unique Smooth Solution to the Cauchy Problem)**.** *Given $f, g \in \mathcal{S}(\mathbb{R}^n)$, there exists a unique solution $\phi \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ of the Cauchy problem, where ϕ is Schwarz in ξ.*

While it is difficult to see the domain of dependence from the above representation formula, we can obtain a conservation law.

Proposition 2.1 (Conservation Law for Solutions to the Cauchy Problem)**.** *Let ϕ be the unique smooth solution to the Cauchy wave problem with appropriate initial data. Then*

$$\|\phi_t\|_{H^{s-1}} + \|\nabla\phi\|_{H^{s-1}} = \|g\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}}.$$

Remark 2.1. *Note that the right side of the expression denotes the original energy of the solution, while the left side denotes the energy at some time $t > 0$. Thus the above relation tells us that the energy of ϕ is conserved. (cf. Problem 1-3.)*

Proof. We start with the representation formula for ϕ in the frequency domain,

$$\hat{\phi}(\xi,t) = \frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi t|\xi|) + \hat{f}(\xi)\cos(2\pi t|\xi|).$$

Differentiating both sides, one obtains

$$\hat{\phi}_t = \hat{g}(\xi)\cos(2\pi t|\xi|) - 2\pi|\xi|\hat{f}(\xi)\sin(2\pi t|\xi|),$$

so that

$$|\hat{\phi}_t|^2 = |\hat{g}(\xi)|^2\cos^2(2\pi t|\xi|) - 4\pi|\xi|\hat{g}(\xi)\hat{f}(\xi)\cos(2\pi t|\xi|)\sin(2\pi t|\xi|) + 4\pi^2|\xi|^2|\hat{f}(\xi)|^2\sin^2(2\pi t|\xi|).$$
$$4\pi^2|\xi|^2|\hat{\phi}|^2 = |\hat{g}(\xi)|^2\sin^2(2\pi t|\xi|) + 4\pi|\xi|\hat{g}(\xi)\hat{f}(\xi)\sin(2\pi t|\xi|)\cos(2\pi t|\xi|) + 4\pi^2|\xi|^2|\hat{f}(\xi)|^2\cos^2(2\pi t|\xi|).$$

Adding these expressions, one obtains

$$|\hat{\phi}_t|^2 + 4\pi^2|\xi|^2|\hat{\phi}|^2 = |\hat{g}(\xi)|^2 + 4\pi^2|\xi|^2|\hat{f}(\xi)|^2.$$

Multiplying both sides by $(1+|\xi|^2)^{s-1}$, integrating, and using the definition of the Sobolev norms (in the frequency domain), one obtains:

$$\left\|\hat{\phi}_t\right\|_{H^{s-1}} + \left\|4\pi^2|\xi|^2\hat{\phi}\right\|_{H^{s-1}} = \|\hat{g}\|_{H^{s-1}} + \left\|4\pi^2|\xi|^2\hat{f}\right\|_{H^{s-1}}.$$

Using Plancherel's theorem, we conclude that

$$\|\phi_t\|_{H^{s-1}} + \|\nabla\phi\|_{H^{s-1}} = \|g\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}}.$$

□

Now using the Fourier representation, we will prove the decay property for solutions to the wave equation.

Proposition 2.2 (Local Decay Rate I)**.** *Let $f,g \in \mathcal{S}(\mathbb{R}^n)$, and ϕ be the unique solution for the Cauchy wave problem. Fix $R>0$. Then there exists a constant $C(f,g,R)>0$ such that for all $|x|\leq R$ and $t\geq 0$,*

$$\sup_{x\in\mathbb{R}^n}|\phi(t,x)| \leq \frac{C}{(1+t)^{n-1}}. \tag{2.8}$$

Proof. We start by writing,

$$\phi(t,x) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)}\left[e^{2\pi it\xi}\left(\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2\pi i|\xi|}\right) + e^{-2\pi it|\xi|}\left(\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2\pi i|\xi|}\right)\right] d\xi.$$

For the sake of this proof, we consider only the case $n=3$, and just the second term in the integral above. Define

$$I := \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x\cdot\xi)}\frac{\hat{g}(\xi)}{2\pi i|\xi|}\, d\xi.$$

Furthermore, we will be considering just the case $t\geq 1$. Introduce polar coordinates on $\mathbb{R}^3$ as follows: let $(|\xi|,\xi_\theta,\xi_\phi)$ be the polar coordinates on $\mathbb{R}^3_\xi$ with $\xi_1 = |\xi|\sin\xi_\theta\cos\xi_\phi$, $\xi_2 = |\xi|\sin\xi_\theta\sin\xi_\phi$, and $\xi_3 = |\xi|\cos\xi_\theta$. The corresponding volume form is,

$$d\xi = |\xi|^2\sin\xi_\theta\, d|\xi|d\xi_\theta d\xi_\phi.$$

Furthermore, one has that for any $N\in\mathbb{N}_0$,

$$\left(\frac{1}{2\pi it}\frac{\partial}{\partial|\xi|}\right)^N e^{2\pi it|\xi|} = e^{2\pi it|\xi|}.$$

Now using integration by parts,

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{2\pi it|\xi|}e^{2\pi i(x\cdot\xi)}\frac{\hat{\phi}_1(\xi)}{2\pi i}|\xi|\sin(\xi_\theta)\, d|\xi|d\xi_\theta d\xi_\varphi \\
&= \left[\frac{1}{(2\pi i)^2 t}\int_0^{\pi}\int_0^{2\pi} e^{2\pi it|\xi|}e^{2\pi i(x\cdot\xi)}\hat{\phi}_1(\xi)|\xi|\sin(\xi_\theta)\, d\xi_\varphi d\xi_\theta\right]_{|\xi|=0}^{R\to\infty} \\
&\quad - \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{1}{2\pi it}e^{2\pi it|\xi|}\frac{\partial}{\partial|\xi|}\left(e^{2\pi ix\cdot\xi}\frac{\hat{\phi}_1(\xi)}{2\pi i}|\xi|\right)\sin(\xi_\theta)\, d|\xi|d\xi_\theta d\xi_\varphi
\end{aligned}$$

We claim that the first term is identically zero. To prove this, we first note that

$$\lim_{|\xi|\to\infty}\left|^{2\pi it|\xi|}e^{2\pi i(x\cdot\xi)}\hat{\phi}_1(\xi)|\xi|\sin(\xi_\theta)\right| = \lim_{|\xi|\to\infty}|\hat{\phi}_1(\xi)||\xi| = 0,$$

since $\hat{\phi}_1(\xi)\in\mathcal{S}(\mathbb{R}^3)$. On the other hand, the integrand vanishes identically at $|\xi|=0$. Now, we see that one application of integration by parts gave us a power of t. To get another power of t, we will use integration by parts on the second term. This time, after dealing with a boundary term at $|\xi|=0$, we get

$$I = -\int_{\mathbb{R}^3} e^{2\pi it|\xi|}\frac{1}{8\pi^3 it^2}(\cdots)\sin(\xi_\theta)\, d|\xi|d\xi_\theta d\theta_\varphi + \frac{1}{2\pi^2 it^2}\hat{\phi}_1(\xi=0).$$

The boundary term does not allow us to use integration by parts again. Thus bounding each of the term and using the fact that $\hat{\phi}_1 \in \mathcal{S}$, one obtains

$$|I| \leq \frac{C(1+R^2)}{t^2}.$$

Note that this decay bound is not uniform for all $x \in \mathbb{R}^n$. ☐

However, we can get a *weaker uniform* decay rate.

Theorem 2.3 (Uniform Decay Rate)**.** *For $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$, and ϕ the unique solution to the homogeneous initial value problem, there exists $C = C(\phi_0, \phi_1) > 0$ so that*

$$\sup_{x \in \mathbb{R}^n} |\phi(t,x)| \leq \frac{C}{(1+t)^{\frac{n-1}{2}}}$$

for $t \geq 0$.

2.3.2 Sobolev Regularity and Generalized Solutions

Till now, we have been assuming that our initial data has strong regularity (recall that the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is a subset of $C^\infty(\mathbb{R}^n)$). But in practice, we might often have initial data with lower regularity. We will now use the theory developed above to derive lower regularity (and generalized) solutions to the Cauchy wave problem. We begin with the following theorem.

Theorem 2.4 (Existence and Uniqueness for Sobolev Regularity Initial Data)**.** *Let $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. Then there exists a unique solution ϕ to the wave equation with $\phi \in C^1(\mathbb{R}, L^2(\mathbb{R}^n)) \cap C^0(\mathbb{R}, H^1(\mathbb{R}^n))$. I.e, ϕ is continuous, and in L^2 and H^1 in the spatial coordinate, and ϕ' is continuous, and L^2 in the spatial coordinate.*

Proof. (Existence): Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there exist sequences $\{f_n\}$, $\{g_n\}$ of smooth compactly supported functions that tend to f and g in $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively. Using the well-posedness for the linear wave equation for smooth data, we obtain a sequence $\{\phi_n\}$ of smooth solutions of the wave equation on $\mathbb{R} \times \mathbb{R}^n$. Now recall the energy functional defined in Section 1.3: for any solution w to the wave equation,

$$2E[w](t) = \int_{\mathbb{R}^n} |\nabla w(t,x)|^2 + |\partial_t w(t,x)|^2 \, dx =: \|w\|^2_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t w\|^2_{L^2(\mathbb{R}^n)},$$

where $\dot{H}^1(\mathbb{R}^n)$ denotes the space of functions whose weak derivatives are in $L^2(\mathbb{R}^n)$ (this space is equipped with the seminorm $\|\nabla f\|_{L^2(\mathbb{R}^n)}$. By energy conservation, for any $T > 0$, one has that

$$\sup_{t \in [-T,T]} \left(\|w(t,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t w(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \right) = \|w(0,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t w(0,\cdot)\|^2_{L^2(\mathbb{R}^n)}.$$

Since the norms $\|\cdot\|^2_{L^2} + \|\cdot\|^2_{\dot{H}^1}$ and $\|\cdot\|_{L^2} + \|\cdot\|_{\dot{H}^1}$ are topologically equivalent, setting $w = \phi_m - \phi_n$, one has that

$$\sup_{t \in [-T,T]} \left(\|\phi_m(t,\cdot) - \phi_n(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t \phi_m(t,\cdot) - \partial_t \phi_n(t,\cdot)\|_{L^2(\mathbb{R}^n)} \right) = \|f_m - f_n\|_{\dot{H}^1(\mathbb{R}^n)} + \|g_m - g_n\|_{L^2(\mathbb{R}^n)}.$$

[!! Complete Later !!] ☐

2.4 Existence and Uniqueness for Linear Wave Equations

Now we will revisit the existence and uniqueness results for general linear wave equations. For context, we will be studying PDE of the following form:

$$\begin{cases} \partial_\alpha (a^{\alpha\beta} \partial_\beta \phi) = F \\ \phi(0,x) = f(x) \\ \partial_t \phi(0,x) = g(x), \end{cases} \tag{2.9}$$

where $a^{\alpha\beta}$ are the components of a $(n+1)\times(n+1)$ symmetric matrix on $\mathbb{I}\times\mathbb{R}^n$ so that

What does this mean, and why can we make this assumption?

$$\sum_{\alpha,\beta}|a^{\alpha\beta}-\eta^{\alpha\beta}|<\frac{1}{10}.$$

Here we are using Einstein summation convention.

2.4.1 Returning to the Energy Estimate

In the following section, we will return to the notion of energy estimates, and eventually use these to derive solutions for general linear wave equations. As a precursor, we make the following notation.

$$|\partial\phi|^2 \coloneqq (\partial_t\phi)^2+\sum_{i=1}^n(\partial_i\phi)^2. \tag{2.10}$$

Moreover, we present **Grönwall's lemma**.

Lemma 2.2 (Grönwall's Lemma)**.** *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonnegative continuous function and $g:\mathbb{R}\to\mathbb{R}$ be a nonnegative integrable function such that*

$$f(t)\le A+\int_0^t f(s)g(s)\,ds$$

holds for some $A\ge 0$, and every $t\in[0,T]$. Then

$$f(t)\le A\exp\left(\int_0^t g(s)\,ds\right)$$

for every $t\in[0,T]$.

Proof.

(1) (Standard Proof): The proof follows by showing that the result holds for any $A>0$, and then passing to the limit $A\to 0$. First, we compute

$$\frac{d}{dt}\left(A+\int_0^t f(s)g(s)\,ds\right)=f(t)g(t)\le g(t)\left(A+\int_0^t f(s)g(s)\,ds\right).$$

In particular, this implies that

$$\frac{d}{dr}\log\left[A+\int_0^t f(s)g(s)\,ds\right]\le g(r),$$

so that integrating,

$$\log\left[A+\int_0^t f(s)g(s)\,ds\right]\le\int_0^t g(r)\,dr+\log A.$$

Thus, we conclude that

$$f(t)\le A+\int_0^t f(s)g(s)\,ds\le A\exp\left(\int_0^t g(r)\,dr\right).$$

(2) (Bootstrap Method): Now we will prove the same result again, using the **bootstrap** method. For each $\varepsilon>0$, consider the following condition:

$$f(t)\le(1+\varepsilon)A\exp\left((1+\varepsilon)\int_0^t g(s)\,ds\right), \tag{$\star$}$$

and define the subset $B\subseteq[0,T]$ by

$$B\coloneqq\{t\in[0,T]:(\star)\text{ holds for all } s\in[0,t]\}.$$

It follows immediately that B is nonempty as it at least contains the point $0 \in B$. Likewise, by continuity of f, it follows that B is closed. By continuity of f, if we can show that f satisfies a *better* estimate for $t \in B$, then it must follow that f satisfies the above bound in some small neighborhood of t, which would show that B is open in $[0,T]$. Indeed, let $t \in B$. Then combining the hypothesis, one has that

$$\begin{aligned} f(t) &\leq A + \int_0^t g(s)\,ds \leq A + (1+\varepsilon)A\int_0^t g(s)\exp\left((1+\varepsilon)\int_0^s g(r)\,dr\right)ds \\ &\leq A\left(1 + \left(\exp\left((1+\varepsilon)\int_0^t g(s)\,ds\right) - 1\right)\right) \qquad \text{(by integration by parts)} \\ &= A\exp\left((1+\varepsilon)\int_0^t g(s)\,ds\right). \end{aligned}$$

Thus, this shows that B must be open. Since $[0,T]$ is connected, one must have $B = [0,T]$. Hence the modified condition holds for all $t \in [0,T]$ and $\varepsilon > 0$ so that passing to the limit $\varepsilon \to 0$ yields the desired estimate.

□

Now we will use Grönwall's lemma to derive an energy estimate for the Cauchy wave problem.

Proposition 2.3 (Energy Estimates for Classical Solutions to Cauchy Problem)**.** *Given a classical solution ϕ to the Cauchy wave problem, 2.12, there exists a constant $C = C(n) > 0$ such that the following energy estimate holds.*

$$\sup_{t\in[0,T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}(t) \leq C\left(\|\nabla f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}\right)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^\infty)}(t)\,dt\right). \quad (2.11)$$

Remark 2.2. *Note that the quantity $\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)$ denotes the total energy of the solution at some time $t \in [0,T]$. The terms,*

$$\|\nabla f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}$$

denote the total initial energy, as well as the energy introduced into the system by the forcing term F. The term

$$\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)$$

denotes the energy introduced into the system due to a possibly dynamic background (e.g., if the background is static so that $\partial a \equiv 0$ for all t, then the exponential term is identically one).

Proof. For the sake of this proof, we will be assuming that ϕ is compactly supported so that we can ignore the spatial boundary terms in the integration. The fundamental idea behind the techniques in this proof is to bound the energy of the solution on the time interval $[0,T]$ by looking at the total energy at some fixed time, and then summing these contributions. In other words, we wish to integrate the first term of the quantity, $\partial_t\phi(\partial_\alpha(a^{\alpha\beta}\partial_\beta\phi) - F)$ by parts. Before doing so, we make the following observations.

(1) Suppose first that $\alpha = \beta = 0$. Then one has that

$$\begin{aligned} \partial_t\phi(\partial_\alpha(a^{\alpha\beta}\partial_\beta\phi)) &= (\partial_t\phi)(\partial_t(a^{tt}\partial_t\phi)) \\ &= (\partial_t\phi)((\partial_t a^{tt})(\partial_t\phi)) + a^{tt}(\partial_t\phi)(\partial_{tt}\phi) \\ &= (\partial_t a^{tt})(\partial_t\phi)^2 + \frac{1}{2}a^{tt}\partial_t(\partial_t\phi)^2, \end{aligned}$$

so that

$$\begin{aligned} \int_0^T\int_{\mathbb{R}^n} (\partial_t\phi)(\partial_t(a^{tt}\partial_t\phi))\,dxdt &= \int_0^T\int_{\mathbb{R}^n} (\partial_t a^{tt})(\partial_t\phi)^2 + \frac{1}{2}a^{tt}\partial_t(\partial_t\phi)^2\,dxdt \\ &= \frac{1}{2}\int_{\{T\}\times\mathbb{R}^n} a^{tt}(\partial_t\phi)^2\,dx - \frac{1}{2}\int_{\{0\}\times\mathbb{R}^n} a^{tt}(\partial_t\phi)^2\,dx \\ &\quad + \frac{1}{2}\int_0^T\int_{\mathbb{R}^n} (\partial_t a^{tt})(\partial_t\phi)^2\,dxdt, \end{aligned}$$

where the second line follows from Fubini/Tonelli, and the fundamental theorem of calculus.

(2) Now assume that one of the indices is t, while the other index runs from $j = 1, \ldots, n$. Then one has that

$$\begin{aligned}\partial_t\phi(\partial_\alpha(a^{\alpha\beta}\partial_\beta\phi)) &= (\partial_t\phi)\cdot\left[\partial_j(a^{jt}\partial_t\phi) + \partial_t(a^{tj}\partial_j\phi)\right]\\ &= (\partial_t\phi)\cdot\left[(\partial_j a^{jt})(\partial_t\phi) + a^{jt}\partial_j(\partial_t\phi) + \partial_t(a^{jt})(\partial_j\phi) + a^{jt}\partial_j(\partial_t\phi)\right]\\ &= (\partial_j a^{jt})(\partial_t\phi)^2 + a^{jt}\partial_j(\partial_t\phi)^2 + (\partial_t a^{jt})(\partial_t\phi)(\partial_j\phi).\end{aligned}$$

Observe that $a^{jt} = a^{tj}$ by symmetry of a. We first consider the integral of the middle term. Integration by parts tells us that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} a^{jt}\partial_j(\partial_t\phi)^2\,dxdt &= \int_0^T\left[a^{jt}(\partial_t\phi)^2|_{\text{bdr.}} - \int_{\mathbb{R}^n}(\partial_j a^{jt})(\partial_t\phi)^2\,dx\right]dt\\ &= -\int_0^T\int_{\mathbb{R}^n}(\partial_j a^{jt})(\partial_t\phi)^2\,dxdt,\end{aligned}$$

where the boundary terms vanish because of our assumption that ϕ is compactly supported. Thus in fact,

$$\int_0^T\int_{\mathbb{R}^n}\partial_t\phi(\partial_\alpha(a^{\alpha\beta}\partial_\beta\phi))\,dxdt = \int_0^T\int_{\mathbb{R}^n}(\partial_j a^{jt})(\partial_t\phi)^2 - (\partial_j a^{jt})(\partial_t\phi)^2 + (\partial_t a^{jt})(\partial_t\phi)(\partial_j\phi)\,dxdt.$$

(3) Now assume that both indices run over $i, j = 1, \ldots, n$. Then one has that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}\partial_t\phi(\partial_i(a^{ij}\partial_j\phi))\,dxdt &= (\partial_t\phi)(a^{ij}\partial_j\phi)\Big|_{\text{bdr.}} - \int_0^T\int_{\mathbb{R}^n}\partial_t\partial_i\phi a^{ij}\partial_j\phi\,dxdt\\ &= -\frac{1}{2}\int_0^T\int_{\mathbb{R}^n}a^{ij}\partial_t(\partial_i\phi\partial_j\phi)\,dxdt\\ &= -\frac{1}{2}\int_{\{T\}\times\mathbb{R}^n}a^{ij}\partial_i\phi\partial_j\phi\,dx + \frac{1}{2}\int_{\{0\}\times\mathbb{R}^n}a^{ij}\partial_i\phi\partial_j\phi\,dx\\ &\quad + \frac{1}{2}\int_0^T\int_{\mathbb{R}^n}(\partial_t a^{ij}(\partial_i\phi)(\partial_j\phi)\,dxdt,\end{aligned}$$

where the final line follows from the fact that

$$\partial_t(a^{ij}\partial_i\phi\partial_j\phi) - (\partial_t a^{ij})(\partial_i\phi)(\partial_j\phi) = a^{ij}\partial_t(\partial_i\phi\partial_j\phi),$$

we used the fundamental theorem of calculus after using the above fact, and the boundary terms in the first line vanish since ϕ has compact support.

Now we obtain the following result:

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}(\partial_t\phi)(\partial_\alpha(a^{\alpha\beta}\partial_\beta\phi)) = \int_0^T\int_{\mathbb{R}^n}(\partial_t\phi)F\\ \implies &\frac{1}{2}\int_{\{T\}\times\mathbb{R}^n}a^{tt}(\partial_t\phi)^2\,dx - \frac{1}{2}\int_{\{T\}\times\mathbb{R}^n}a^{ij}\partial_i\phi\partial_j\phi\,dx = \frac{1}{2}\int_{\{0\}\times\mathbb{R}^n}a^{tt}(\partial_t\phi)^2\,dx - \frac{1}{2}\int_{\{0\}\times\mathbb{R}^n}a^{ij}\partial_i\phi\partial_j\phi\,dx\\ &\qquad + \int_0^T\int_{\mathbb{R}^n}(\partial_t\phi)F - \frac{1}{2}(\partial_t a^{tt})(\partial_t\phi)^2 - (\partial_j a^{jt})(\partial_t\phi)^2\\ &\qquad\quad + (\partial_j a^{jt})(\partial_t\phi)^2 - (\partial_t a^{jt})(\partial_t\phi)(\partial_j\phi)\,dxdt\end{aligned}$$

Now, one has that

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^n}(\partial_t\phi)F - \frac{1}{2}(\partial_t a^{tt})(\partial_t\phi)^2 - (\partial_j a^{jt})(\partial_t\phi)^2 + (\partial_j a^{jt})(\partial_t\phi)^2 - (\partial_t a^{jt})(\partial_t\phi)(\partial_j\phi)\,dxdt\right|\\ &\qquad \le C'\int_0^T\|\partial_t\phi\|_{L^2(\mathbb{R}^n)}\|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\,dt,\end{aligned}$$

where we used Hölder's inequality to obtain the first term, and assumed $\partial a \in L^\infty(\mathbb{R}^n)$. On the other hand, by our hypothesis on $a^{\alpha\beta}$, there exists some constant $K > 0$ large enough so that (taking $C = \min\{C', K\}$), Explain how?

$$\|\partial\phi\|_{L^2(\mathbb{R}^n)}(T) \le C\left|\frac{1}{2}\int_{\{T\}\times\mathbb{R}^n} a^{tt}(\partial_t\phi)^2\,dx - \frac{1}{2}\int_{\{T\}\times\mathbb{R}^n} a^{ij}\partial_i\phi\partial_j\phi\,dx\right|.$$

Thus this implies that

$$\|\partial\phi\|_{L^2(\mathbb{R}^n)}(T) \le C\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\int_0^T \|\partial_t\phi\|_{L^2(\mathbb{R}^n)}\|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\,dt.$$

Since one can repeat this argument for any smaller interval $[0, T']$, where $0 < T' < T$, one obtains

$$\sup_{t\in[0,T]}\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t) \le C\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\int_0^T \|\partial_t\phi\|_{L^2(\mathbb{R}^n)}\|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\,dt.$$

For the first term inside the integral, one has that

$$\begin{aligned}\int_0^T \|\partial_t\phi\|_{L^2(\mathbb{R}^n)}\|F\|_{L^2(\mathbb{R}^n)}(t)\,dt &\le \sup_{t\in[0,T]}\|\partial_t\phi\|_{L^2(\mathbb{R}^n)}\cdot\left(\int_0^T\|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)\\ &\le \delta\|\partial_t\phi\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{4\delta}\left(\int_0^T\|F\|_{L^2(\mathbb{R}^n)}(t)\,dt\right)^2.\end{aligned}$$

where the second line follows from Young's inequality with ε[3]. If we fix $\delta > 0$ small enough so that the first term can be combined with the left side of the above inequality (and absorbing δ into the constant C), one has that

$$\sup_{t\in[0,T]}\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t) \le C\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\left(\int_0^T\|F\|^2_{L^2(\mathbb{R}^n)}(t)\,dt\right)^2 + C\int_0^T\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\,dt.$$

Thus using Grönwall's lemma concludes the proof. □

We have the following corollary.

Corollary 2.1 (Uniqueness of Classical Solutions to General Linear Wave Equation)**.** *Any classical solution of the general Cauchy wave problem, 2.12, is unique.*

Proof. Let ϕ_1, ϕ_2 be two solutions to the general Cauchy wave problem 2.12. Then their difference $\psi := \phi_1 - \phi_2$ satisfies the Cauchy wave problem

$$\begin{cases}\partial_\alpha(a^{\alpha\beta}\partial_\beta\psi) = 0\\ \psi(0,x) = 0\\ \partial_t\psi(0,x) = 0,\end{cases}$$

Thus using the energy estimate we derived above, one has that for any fixed $T > 0$,

$$\sup_{t\in[0,T]}\|\partial\psi\|_{L^2(\mathbb{R}^n)}(t) = 0,$$

so that the energy of ψ is always zero. This must mean that $\psi_t, \nabla\psi = 0$ a.e. within $[0,T]\times\mathbb{R}^n$. But since $\psi \equiv 0$ on $\{t = 0\}\times\mathbb{R}^n$, we conclude that $\psi = 0 \iff \phi_1 = \phi_2$ a.e. on $[0,T]\times\mathbb{R}^n$. Since T was arbitrary, the proof concludes. □

[3]For any $a, b \ge 0$ and any $\varepsilon > 0$,

$$ab \le \frac{\varepsilon^p a^p}{p} + \frac{1}{e^q q}b^q$$

, where p, q are conjugate exponents.

Now we will derive some higher order energy estimates. In particular, we will show that the higher order H^k norms are also propagated.

Proposition 2.4 (Higher Order Energy Estimate I)**.** *Let ϕ be a smooth solution to the Cauchy wave problem 1.21, and let k be a positive integer. Then for some constant $C = C(n,k) > 0$, the following energy estimates holds:*

$$\begin{aligned}\sup_{t\in[0,T]} \|\phi\|_{H^k(\mathbb{R}^n)}(t) &+ \sup_{t\in[0,T]} \|\partial_t\phi\|_{H^{k-1}(\mathbb{R}^n)}(t) \\ &\le C(1+T)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\,dt\right)\Bigg(\|f\|_{H^k(\mathbb{R}^n)} + \|g\|_{H^{k-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t)\,dt \\ &\qquad + C\int_0^T \sum_{|\alpha|+|\beta|\le k-1} \left\|\partial\partial_x^\alpha a\partial\partial_x^\beta\phi\right\|_{L^2(\mathbb{R}^n)}(t) + \sum_{|\alpha|+|\beta|\le k-1} \left\|\partial_x^\alpha a\partial\partial_x^\beta\phi\right\|_{L^2(\mathbb{R}^n)}(t)\,dt\Bigg).\end{aligned} \tag{2.12}$$

Remark 2.3. *Notice that in this bound, we are using the Sobolev space, $H^k(\mathbb{R}^n)$, norms. As we discussed in Section 2.2, these norms measures the "L^2 size" of ϕ, $\partial_t\phi$ and their spatial derivatives up to order k and $k-1$, respectively. Therefore, the above estimate tells us something about the higher-frequency energy of the solution. Another interesting feature is the difference in regularity that our estimate has for each term (notably, we only require $k-1$-regularity on our source, but k regularity for ϕ and the initial data f).*

Proof. [!! Complete Later !!] □

2.4.2 Existence of Solutions

In the section above, we *assumed* that the Cauchy wave problem had a classical solution, and deduced various energy estimates such a solution must satisfy. In particular, we showed that *if* a classical solution exists, then it must be unique. But we are yet to show the existence of a solution. In this section, we will use elements of functional analysis to prove the following existence theorem.

Theorem 2.5 (Existence of Unique Solution to General Cauchy Wave Problem)**.** *Let k be a fixed positive integer. Given $(f,g)\in H^k(\mathbb{R}^n)\times H^{k-1}(\mathbb{R}^n)$, and $F\in L^1([0,T],H^{k-1}(\mathbb{R}^n))$, there exists a unique solution*

$$\phi\in C^0([0,T],H^k(\mathbb{R}^n))\cap C^1([0,T],H^{k-1}(\mathbb{R}^n))$$

to the general Cauchy wave problem 2.12.

The main tools we will use to prove this result are the **Hahn-Banach theorem** (which we will state below), the energy estimates from the previous section, and the **Sobolev embedding theorem**.

Theorem 2.6 (Hahn-Banach)**.** *Let ϕ be a bounded linear functional*

A Proof of Theorem 2.1

Proof. (1) First assume that $f, \widehat{f}, g, \widehat{g} \in L^1(\mathbb{R}^n)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)\, d\xi &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx \right) g(\xi)\, d\xi \\
&= \int_{\mathbb{R}^n} f(x) \cdot \left(\int_{\mathbb{R}^n} g(\xi) e^{-2\pi i x\cdot\xi}\, d\xi \right) dx \qquad \text{(by Fubini/Tonelli)} \\
&= \int_{\mathbb{R}^n} f(x)\widehat{g}(x)\, dx.
\end{aligned}$$

Pointwise, it is clear that

$$\lim_{\varepsilon\to 0} e^{-\varepsilon\pi|\xi|^2 + 2\pi i x\cdot\xi} \widehat{f}(\xi) = e^{2\pi i x\cdot\xi} \cdot \widehat{f}(\xi)$$

for a.e. $\xi \in \mathbb{R}^n$. On the other hand, we note that

$$\left| e^{-\varepsilon\pi|\xi|^2} e^{2\pi i x\cdot\xi} \widehat{f}(\xi) \right| = \left| e^{-\varepsilon\pi|\xi|^2} \right| \cdot \left| \widehat{f}(\xi) \right| \le \left| \widehat{f}(\xi) \right| \in L^1(\mathbb{R}^n),$$

for any $\varepsilon > 0$. Thus, by the Dominated Convergence Theorem, we have

$$(\widehat{f})^{\vee} = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{-\varepsilon\pi|\xi|^2 + 2\pi i x\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

Let $g(\xi) = e^{-\varepsilon^2\pi|\xi|^2 + 2\pi i x\cdot\xi}$. First, by definition of the Fourier transform, one has

$$\begin{aligned}
\widehat{g}(y) &= \int_{\mathbb{R}^n} g(\xi) e^{-2\pi i \xi\cdot y}\, d\xi \\
&= \int_{\mathbb{R}^n} e^{-\varepsilon^2\pi|\xi|^2} e^{2\pi i \xi\cdot(x-y)}\, d\xi \\
&= \int_{\mathbb{R}^n} e^{-\varepsilon^2\pi\left(|\xi|^2 - \frac{2i}{\varepsilon^2}\xi\cdot(x-y) - \frac{1}{\varepsilon^4}|x-y|^2\right)} e^{-\frac{\pi|x-y|^2}{\varepsilon^2}}\, d\xi \\
&= e^{-\frac{\pi|x-y|^2}{\varepsilon^2}} \cdot \int_{\mathbb{R}^n} e^{-\varepsilon^2\pi|\xi - (i/\varepsilon^2)(x-y)|^2}\, d\xi \\
&= \varepsilon^{-n} e^{-\frac{\pi|x-y|^2}{\varepsilon^2}}.
\end{aligned}$$

Let ϕ be a compactly supported smooth function such that $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$, and let $\beta \in L^p(\mathbb{R}^n)$. We will show the equivalent result that $\beta_\varepsilon(x) := \int_{\mathbb{R}^n} \beta(x-y)\varepsilon^{-n}\phi\left(\frac{y}{\varepsilon}\right) dy$ converges to $\beta(x)$ in L^p as $\varepsilon \to 0$. Further assume $\phi \ge 0$. Then we compute

$$\begin{aligned}
\|\beta_\varepsilon - \beta\|_p^p &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} [\beta(x-y) - \beta(x)]\, \varepsilon^{-n}\phi\left(\frac{y}{\varepsilon}\right) dy \right|^p dx \\
&\le \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |\beta(x-y) - \beta(x)|\, \varepsilon^{-n}\phi\left(\frac{y}{\varepsilon}\right) dy \right]^p dx \\
&\left(\le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\beta(x-y) - \beta(x)|^p\, dx \right)^{1/p} \varepsilon^{-n}\phi\left(\frac{y}{\varepsilon}\right) dy \right)^p \qquad \text{(by Minkowski's Integral Inequality)} \\
&\le \left(\int_{\mathbb{R}^n} 2\, \|\beta\|_p\, \varepsilon^{-n}\phi\left(\frac{y}{\varepsilon}\right) dy \right)^p.
\end{aligned}$$

Since the integrand tends to 0 a.e. as $\varepsilon \to 0$, and is dominated by a suitable L^1 function, we conclude that $\beta_\varepsilon \to \beta$ under the L^p norm as $\varepsilon \to 0$. We shall now conclude the proof of the Fourier Inversion Formula as follows:

$$\begin{aligned}
(\widehat{f})^{\vee} &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} g(\xi)\widehat{f}(\xi)\, d\xi \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(y)\widehat{g}(y)\, dy \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} f(y)\varepsilon^{-n} e^{-\frac{\pi|x-y|^2}{\varepsilon^2}}\, dy \\
&\xrightarrow[\text{in } L^p]{\varepsilon\to 0} f(x).
\end{aligned}$$

Thus, $(\widehat{f})^{\vee} = f(x)$ a.e.

(3) We consider

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 e^{-\pi\varepsilon|\xi|^2}\, d\xi &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x)e^{-2\pi i(x,\xi)}\, dx\right)\cdot\left(\int_{\mathbb{R}^n} \overline{f(y)}e^{2\pi i(y,\xi)}\, dy\right) e^{-\pi\varepsilon|\xi|^2}\, d\xi \\ &= \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x)\overline{f(y)}\, dxdy\right)\cdot\int_{\mathbb{R}^n} e^{-\pi\varepsilon|\xi|^2+2\pi i(\xi,y-x)}d\xi \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x)\overline{f(y)}\varepsilon^{-n/2}e^{-\frac{\pi|x-y|^2}{\varepsilon}}\, dxdy \\ &\xrightarrow{\varepsilon\to 0} \iint_{\mathbb{R}^n\times\mathbb{R}^n} f(x)\overline{f_\varepsilon(x)}\, dx.\end{aligned} \tag{A.1}$$

Since $f_\varepsilon(x)\to f$ in L^2, the RHS tends to $\int_{\mathbb{R}^n}|f|^2\, dx$. On the other hand, the left side tends to $\int_{\mathbb{R}^n}|\hat{f}|^2\, d\xi$ by the monotone convergence theorem. Hence, the proof concludes.

□