When Mathematics Meets Astrophysics

Analyzing the Number of Solutions to the Gravitational Lens Equation

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March 31, 2025

Outline

A math problem

Let $g \geqslant 1$ be a positive integer, and $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_g$ be g singularities in a plane with "masses" $m_1, \ldots, m_g$, respectively. Consider the 2-d equation:

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \sum_{j=1}^{g} m_j \frac{\boldsymbol{\theta} - \boldsymbol{\xi}_j}{|\boldsymbol{\theta} - \boldsymbol{\xi}_j|^2}. \quad (1)$$

Question:

How many solutions, $\boldsymbol{\theta}$, exist for a given $\boldsymbol{\beta}$? (*Hint*: Answer in terms of g.)

Gravitational lensing

Eq. (1) comes from gravitational lensing, which can be visualized as:

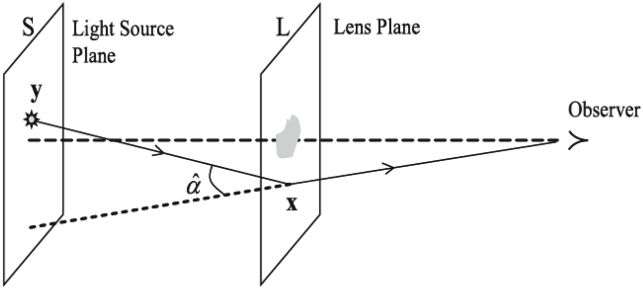


Figure 1: Depiction of gravitational lensing. For future reference, $\boldsymbol{\beta} := \mathbf{y}$ and $\boldsymbol{\theta} := \mathbf{x}$. This figure is borrowed from Petters & Werner (2010).

Theorem (Petters 1992)

Let $g \geqslant 0$ be the infinite singularities in a potential $\psi : L \to \mathbb{R}$. Suppose $\mathbf{y} = \boldsymbol{\beta}(\boldsymbol{\theta})$ is not on a *caustic*[a], and $\boldsymbol{\beta}(\boldsymbol{\theta})$ is *locally stable*[b] with $\text{crit}(\boldsymbol{\beta})$ bounded. Then for sufficiently large $\mathbf{y}$, the lower bounds on the number of solutions to Eq. (1) are attained: $N = g + 1$.

[a] $\text{caus}(\boldsymbol{\beta}) := \{\boldsymbol{\beta}(\boldsymbol{\theta}) \in S : \det[\text{Jac}(\boldsymbol{\beta})](\boldsymbol{\theta}) = 0\}$; The set of points where the magnification formally diverges.

[b] $\text{crit}(\boldsymbol{\beta})$ consists of only folds and cusps.

Theorem (Rhie 2003, Khavinson & Neumann 2005)

Let there be $g \geqslant 2$ infinite singularities in a potential $\psi : L \to \mathbb{R}$ and suppose $\mathbf{y} = \boldsymbol{\beta}(\boldsymbol{\theta})$ is not on a caustic. Then the number of solutions to Eq. (1) satisfies: $N \leqslant 5g - 5$.

Altogether, $g + 1 \leqslant N \leqslant 5g - 5$ **solutions!**

Mathematical Framework

- Proving the lower bound requires differential topology and Morse Theory.
  - From *Fermat's Principle*, solutions exist at the critical points of the *time-delay* surface $T_{\boldsymbol{\beta}} : L \to \mathbb{R}$,

$$\nabla T_{\boldsymbol{\beta}} = \mathbf{0} = -\boldsymbol{\beta} + \boldsymbol{\theta} - \nabla \psi := -\boldsymbol{\beta} + \boldsymbol{\theta} - \frac{d_{LS}}{d_S} \hat{\boldsymbol{\alpha}}. \quad (2)$$

  - Bounds on the number of such points in the surface are given by the *Betti numbers*, B_0, B_1, B_2[1], which are *topological invariants* of a surface.
  - This proof relies on the time-delay surface (and hence on $\hat{\boldsymbol{\alpha}}$) diverging at each m_j.
- Proving the upper bound requires elements of complex analysis and harmonic analysis.

[1] B_k can be thought of as the "number" of k-dimensional holes in the surface.

Applications to Astrophysics

- Eqs. (1) and (2), written differently below, are the *small-angle gravitational lens equation*.

$$\underbrace{\boldsymbol{\beta}}_{\substack{\text{src.}\\\text{pos.}}} = \underbrace{\boldsymbol{\theta}}_{\substack{\text{img.}\\\text{pos.}}} - \underbrace{\frac{d_{LS}}{d_S}\hat{\boldsymbol{\alpha}}(\boldsymbol{\theta})}_{\text{deflection angle}} \quad . \tag{3}$$

- *Gravitational lensing* is the relativistic phenomenon in which spacetime curvature (e.g., due to mass) causes light rays to "bend" around a mass.

Applications to Astrophysics

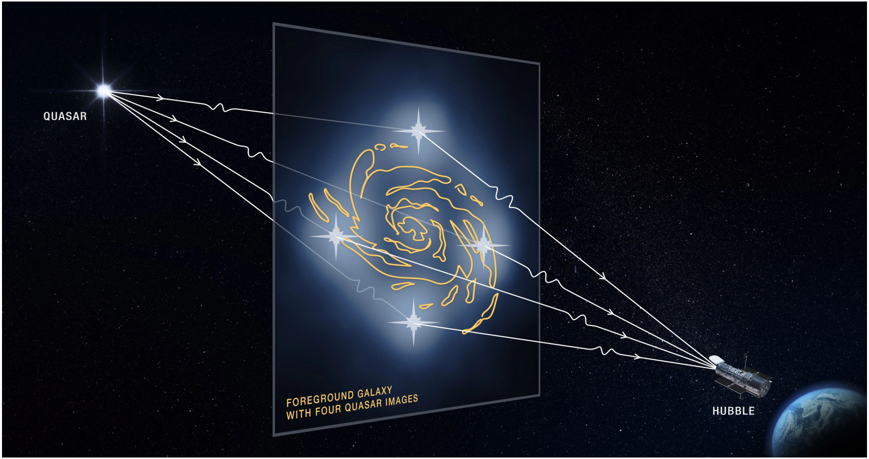


Figure 2: Schematic depiction of gravitational lensing. The source (quasar) is seen by the observer at angular position β. There are five images (i.e., five solutions to Eq. 3). **Credit:** NASA, ESA, and D. Player (STScI)

Motivation

- Accounting for the total number of lensed images in observations is crucial for modeling the lens system, and accurately and precisely reconstructing the properties of the source
  - Members of our lensing group work on **strong lensing** in which multiple galaxies created complicated configurations of multiple images, and on **microlensing** where crossing complicated patterns of caustics due to the "granular" distribution of stars can add or substract number of lensed images.
- Mathematically, variations of this equation lead to interesting questions (e.g., how many solutions do "rational harmonic functions" have?; Khavinson & Neumann 2005) and unifies various mathematical subfields like differential topology and complex analysis.

Our Work

- We studied the *full-angle* **Virbhadra-Ellis Lens Equation** (Virbhadra & Ellis 2000):

$$\tan\left(\boldsymbol{\beta}(\boldsymbol{\theta})\right) = \tan\left(\boldsymbol{\theta}\right) - \frac{d_{LS}}{d_S}\left[\tan\left(\boldsymbol{\theta}\right) + \tan\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\theta}\right)\right], \tag{4}$$

where $\hat{\alpha}$ is the deflection in the **Newtonian** gravitational framework.

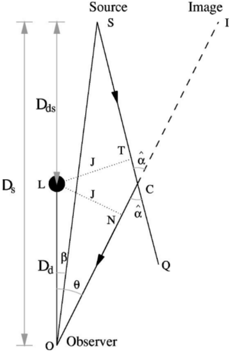


Figure 3: Schematic depiction of the lensing geometry. This figure is borrowed from Virbhadra & Ellis (2000).

Our Work

- For a system of g masses, $\hat{\alpha}$ is found numerically by solving a differential equation for the trajectory of a light ray and finding the difference between the initial and final slopes.

$$\frac{d^2\mathbf{r}}{dt^2} = -G\sum_{j=1}^{g} m_j \frac{\mathbf{r}-\boldsymbol{\xi}_j}{|\mathbf{r}-\boldsymbol{\xi}_j|^3} \tag{5}$$

- We are interested in seeing if the bounds found by Petters (1992), Rhie (2003), and Khavinson & Neumann (2005) hold for our analysis.

Our Work in Context of Previous Research

	Previous Work	Our Work
Newtonian	×	✓
Small-Angle Approximation	✓	×
Born Approximation	✓	×
Thin-Lens Approximations	✓	×
Mathematically Interesting?	✓	✓

Table 1: Table comparing previous work with our work.

- **Small-Angle Approximation:** Angular positions and deflection angles are assumed small ($\ll 1$ rad).
- **Born Approximation:** Deflection is computed by calculating total impulse along **undeflected** light ray.
- **Thin-Lens Approximation:** An incoming light ray bends only as it passes through L.

Methodology

- Derive a closed-form expression for the Newtonian deflection angle for a single point mass. Can the deflection angle diverge at the mass?
- Develop a numerical routine (`deflecThor`) to calculate the deflection angles for g point-masses using Eq. (5).
- Use `deflecThor`, `pygravlens` (routine that finds images for the small-angle GR lens equation; **credit:** Professor Keeton), and 2D root finding to find solutions to Eq. (4).
  - Can we get multiple images? Can we saturate the upper and lower bounds?

Analyzing the Deflection Angle

- Using energy and angular momentum conservation, the (magnitude of the) deflection angle for a *single* mass is,

$$|\hat{\boldsymbol{\alpha}}(b)| = 2\arcsin\left(\frac{\tilde{r}}{\sqrt{\tilde{r}^2 + b^2}}\right), \tag{6}$$

  where b is the impact parameter, and $\tilde{r} = (1/2)r_s = GM/c^2$ is half the "Schwarzschild" radius, r_s.
- We find that $\lim_{b \to 0} |\hat{\boldsymbol{\alpha}}| = \pi$, and $|\hat{\boldsymbol{\alpha}}| < \infty$ for all b.
  - Deflection never diverges!
  - The analysis used by Petters (1992) must be modified.

Analyzing the Deflection Angle

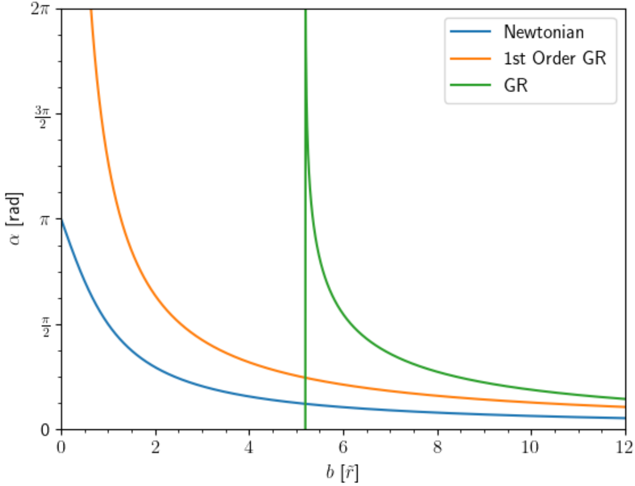


Figure 4: Comparison of Eq. (6), the 1st order GR approximation ($4GM/c^2b$), and the full GR deflection angle. The horizontal axis is expressed in terms of $\tilde{r}$. The latter diverges at $3\sqrt{3}\tilde{r}$.

Analyzing the Deflection Angle

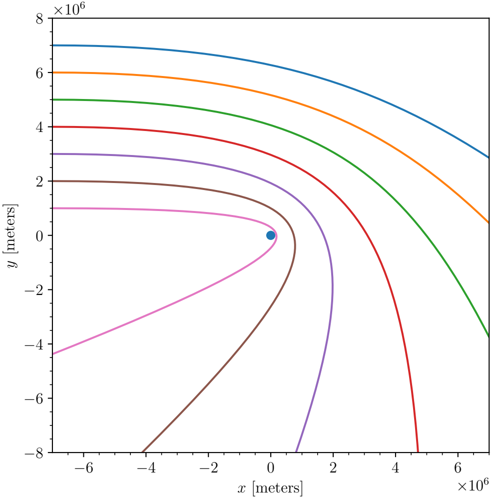


Figure 5: Plots showing light ray trajectories for various impact parameters. Each deflection angle is less than π rad.

Analyzing the Deflection Angle

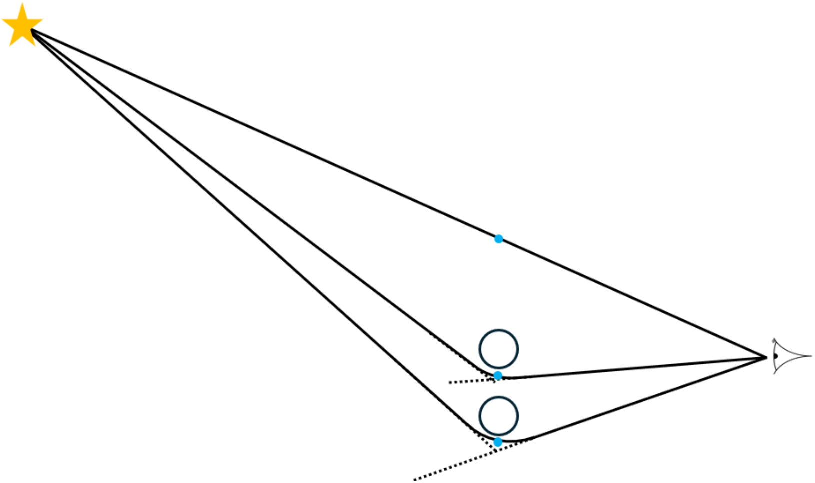


Figure 6: Heuristically, multiple images should be possible. This follows the conditions listed in the first theorem (Petters 1992). All deflection angles are less than π rad, and are achievable.

Analyzing the Number of Images: $g = 1$:

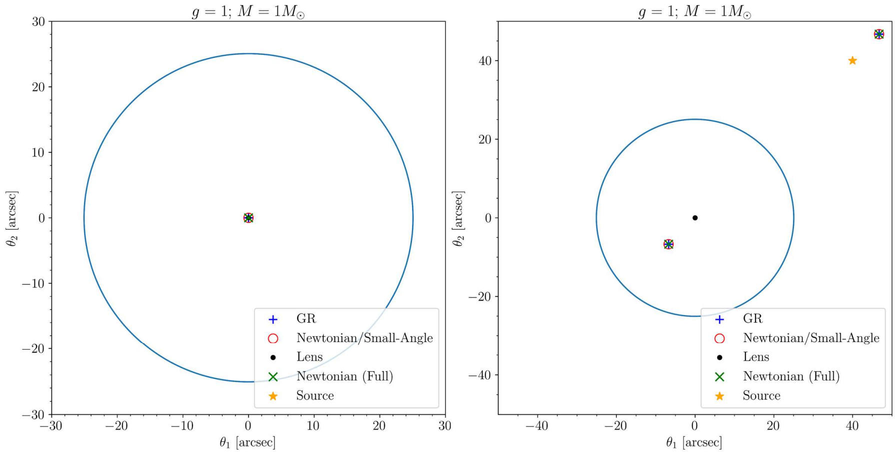
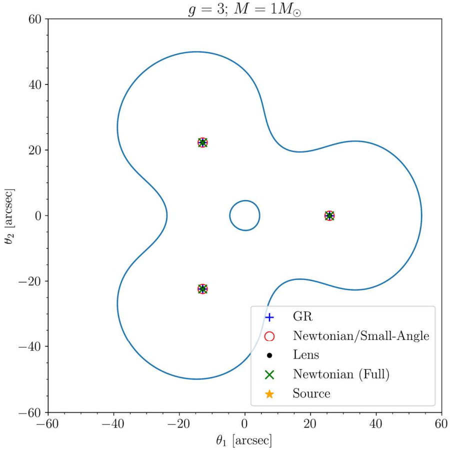


Figure 7: $g = 1$, $d_{LS} = d_L = 10^{11}$ m. **Left:** Saturating the lower bound (2) images with $\boldsymbol{\beta} = (10^5, 0)''$. **Right:** Three images with $\boldsymbol{\beta} = (40, 40)''$.

Analyzing the Number of Images: $g = 3$:



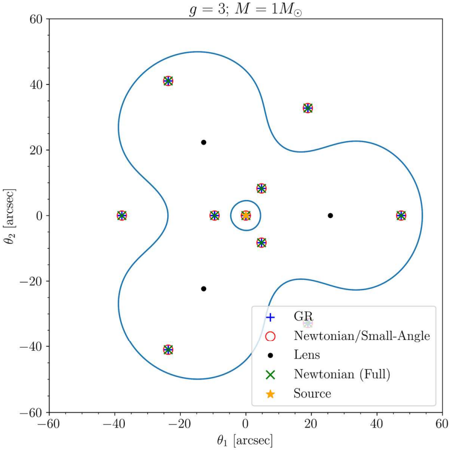


Figure 8: $g = 3$, $d_{LS} = d_L = 10^{11}$ m. **Left:** Saturating the lower bound (4) images with $\boldsymbol{\beta} = (10^5, 0)''$. **Right:** Saturating the upper bound (10) with $\boldsymbol{\beta} = (0, 0)''$.

Discussion

- Our lens equation Eq. (4) can have multiple distinct solutions.
- Image positions are generally consistent with image positions found for the small-angle GR and small-angle Newtonian equations.
  - Average difference between full-angle positions and small-angle positions is $\sim 10^{-3}$".
  - Most likely not a numerical error since varying the 2D root finder/`deflecThor` precision does not affect this difference.

Conjecture

Let N be the number of solutions to Eq. (4) for a system consisting of g point masses in a single plane. We conjecture that N satisfies the same bounds as found by Petters (1992), Rhie (2003), and Khavinson & Neumann (2005).

Conclusion

- We studied the Virbhadra-Ellis lens equation with Newtonian deflection angle $\hat{\alpha}$ (which we found numerically):

$$\tan\left(\beta(\boldsymbol{\theta})\right) = \tan\left(\boldsymbol{\theta}\right) - \frac{d_{LS}}{d_S}\left[\tan\left(\boldsymbol{\theta}\right) + \tan\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\theta}\right)\right]$$

- We found multiple solutions to this equation, and conjectured that this equation satisfies the same bounds on the number of images as the simplified equation.
- To prove this conjecture, we need to revisit the original differential topology and analysis proofs. This could lead to new and interesting mathematical connections.
- In the future, we can explore multiplane lensing (e.g., Keeton et al. 2023), and the Virbhadra-Ellis equation in the GR framework.
- More broadly, we can explore the rich interactions between mathematics and astrophysics.

References

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Thank you!